

# Around Brouwer's fixed point theorem

## (Lecture notes)

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### Abstract

These lecture notes were written about 15 years ago, with a history that goes back nearly 30 years for some parts. They can be regarded as a “prequel” to the book [Mat07], and one day they may become a part of a more extensive book project. They are not particularly polished, but we decided to make them public in the hope that they might be useful. We refer to [Mat07] for notation and terminology not explained here.

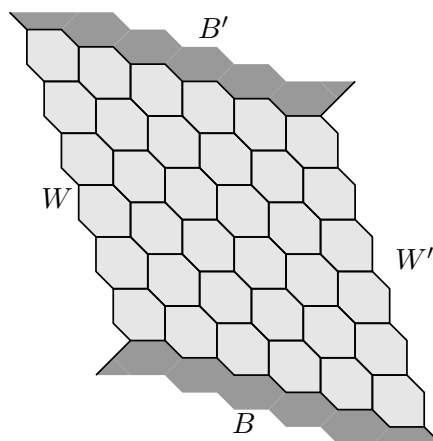
## Contents

<b>1</b>	<b>The Game of HEX and the Brouwer Fixed Point Theorem</b>	<b>2</b>
<b>2</b>	<b>Who wins HEX?</b>	<b>10</b>
<b>3</b>	<b>Piercing multiple intervals</b>	<b>14</b>
<b>4</b>	<b>The fixed point theorems of Lefschetz, Smith, and Oliver</b>	<b>24</b>
<b>5</b>	<b>Evasiveness</b>	<b>29</b>
5.1	A general model . . . . .	29
5.2	Complexity of graph properties . . . . .	29
5.3	Decision trees . . . . .	33
5.4	Monotone systems . . . . .	38
5.5	A topological approach . . . . .	39

# 1 The Game of HEX and the Brouwer Fixed Point Theorem

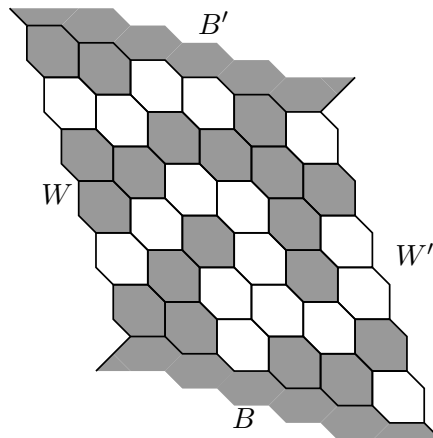
Let's start with a game: "HEX" is a board game for two players, invented by the ingenious Danish poet, designer and engineer Piet Hein in 1942 [Gar89], and rediscovered in 1948 by the mathematician John Nash [Mil95] who got a Nobel prize in economics in 1994 (for his work on game theory, but not really for this game ...).

HEX, in Hein's version, is played on a rhombical board, as depicted in the figure.



The rules of the game are simple: there are two players, whom we call White and Black. The players alternate, with White going first. Each move consists of coloring one "grey" hexagonal tile of the board white resp. black. White has to connect the white borders of the board (marked  $W$  and  $W'$ ) by a path of his white tiles, while Black tries to connect  $B$  and  $B'$  by a black path. They can't both win: any winning path for white separates the two black borders, and conversely. (This isn't hard to prove—however, the statement is closely related to the Jordan curve theorem, which is trickier than it may seem when judged at first sight: see Exercise 8.)

However, here we concentrate on the opposite statement: there is no draw possible—when the whole board is covered by black and white tiles, then there always is a winner. (This is even true if one of the players has cheated badly and ends up with much more tiles than his/her opponent! It is also true if the board isn't really "square," that is, if it has sides of unequal lengths.) Our next figure depicts a final HEX position—sure enough one of the players has won, and the proof of the following "HEX theorem" will give us a systematic method to find out which one.



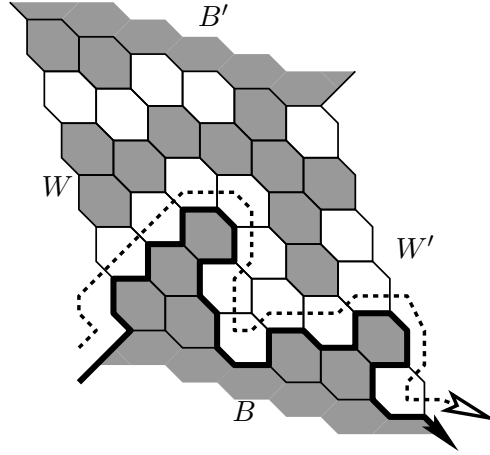
**Theorem 1.1** (The HEX theorem). *If every tile of an  $(n \times m)$ -HEX board is colored black or white, then either there is a path of white tiles that connects the white borders  $W$  and  $W'$ , or there is a path of black tiles that connects the black borders  $B$  and  $B'$ .*

Our plan for this section is the following:

- We give a simple proof of the HEX theorem.
- We show that it implies the Brouwer fixed point theorem,
- And even conversely: the Brouwer fixed point theorem implies the HEX theorem.
- Then we prove that one of the players has a winning strategy.
- And then we see that on a square board, the first player can win, while on an uneven board, the player with the longer borders has a strategy to win.

All of this is really quite simple, but it nicely illustrates how a topological theorem enters the analysis of a discrete situation.

**Proof of the HEX theorem.** We follow a certain path *between* the black and white tiles that starts in the lower left-hand corner of the HEX board on the edge that separates  $W$  and  $B$ . Whenever this path reaches a corner of degree three, there will be both colors present at the corner (due to the edge we reach it from), and so there will be a unique edge to proceed on that does have different colors on its two sides.



Our path can never get stuck or branch or turn back onto itself: otherwise we would have found a vertex that has one or three edges that separate colors, whereas this number clearly has to be even at each vertex. Thus the path can be continued until it leaves the board—that is, until it reaches  $W'$  or  $B'$ . But that means that we find a path that connects  $W$  to  $W'$ , or  $B$  to  $B'$ , and on its sides keeps a white path of tiles resp. a black path. That is, one of White and Black has won! 🐼

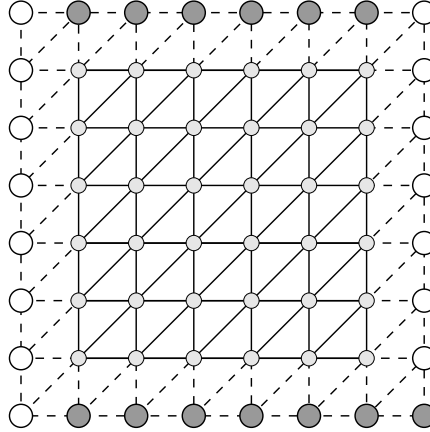
Now this was easy, and (hopefully) fun. We continue with a re-interpretation of the HEX board—in Nash’s version—that buys us two drinks for the price of one:

- (i) a  $d$ -dimensional version of the HEX theorem, and
- (ii) the connection to the Brouwer fixed point theorem.

**Definition 1.2** (The  $d$ -dimensional HEX board). The  $d$ -dimensional *HEX board* is the graph  $H(n, d)$  with vertex set  $V = \{-1, 0, 1, \dots, n, n+1\}^d$ , in which two vertices  $\mathbf{v}, \mathbf{w} \in V$  are connected if  $\mathbf{v} - \mathbf{w} \in \{0, 1\}^d \cup \{0, -1\}^d$ .

The *colors* for the  $d$ -dimensional HEX game are  $1, 2, \dots, d$ , where we identify “1 = white” and “2 = black.” The *interior* of the HEX board is given by  $V' = \{0, 1, 2, \dots, n\}^d$ . All the other vertices, in  $V \setminus V'$ , form the *boundary* of the board. The vertices in the boundary of  $H(n, d)$  get preassigned colors

$$\kappa(v_1, \dots, v_d) := \begin{cases} \min\{i : v_i = -1\} & \text{if this exists,} \\ \min\{i : v_i = n+1\} & \text{otherwise.} \end{cases}$$



Our drawing depicts the 2-dimensional HEX board  $H(5, 2)$ , which represents a dual graph for the  $(6 \times 6)$ -board that we used in our previous figures, with the preassigned colors on the boundary.

The  $d$ -dimensional HEX game is played between  $d$  players who take turns in coloring the interior vertices of  $H(n, d)$ . The  $i$ -th player *wins* if he<sup>1</sup> achieves a path of vertices of color  $i$  that connects a vertex whose  $i$ -th coordinate is  $-1$  to a vertex whose  $i$ -th coordinate is  $n + 1$ .

**Theorem 1.3** (The  $d$ -dimensional HEX theorem). *There is no draw possible for  $d$ -dimensional HEX: if all interior vertices of  $H(d, n)$  are colored, then at least one player has won.*

**Proof.** The proof that we used for 2-dimensional HEX still works: it just has to be properly translated for the new setting. For this we first check that  $H(n, d)$  is the graph of a triangulation  $\Delta(n, d)$  of  $[-1, n + 1]^d$ , which is given by the *clique complex* of  $H(n, d)$ : that is, a set of lattice points  $S \subseteq \{-1, 0, 1, \dots, n + 1\}^d$  forms a simplex in  $\Delta(n, d)$  if and only if the points in  $S$  are pairwise connected by edges. (To check this, verify that each point  $x \in [-1, n + 1]^d$  lies in the relative interior of a unique simplex, which is given by

$$\begin{aligned} \Delta(x) := \text{conv}\{v \in \{-1, \dots, n + 1\}^d : \\ [x_i] \leq v_i \leq \lceil x_i \rceil \text{ for all } i, \\ [x_i - x_j] \leq v_i - v_j \leq \lceil x_i - x_j \rceil \text{ for all } i \neq j\}. \end{aligned}$$

Our picture illustrates the 2-dimensional case.)

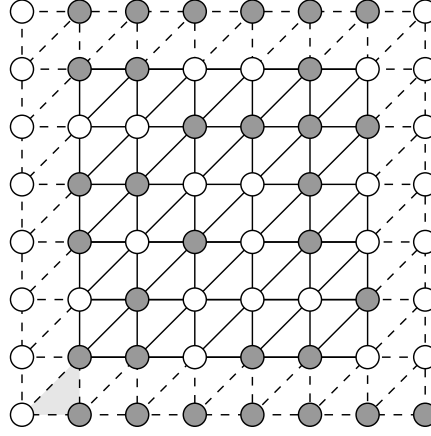
Now every full-dimensional simplex in  $\Delta(n, d)$  has  $d + 1$  vertices. A simplex  $S$  in  $\Delta(n, d)$  is *completely colored* if it has all  $d$  colors on its vertices. Thus each completely colored  $d$ -simplex in  $\Delta$  has exactly two completely colored facets, which are  $(d - 1)$ -faces of the complex  $\Delta(n, d)$ . Conversely, every completely colored  $(d - 1)$ -face is contained in exactly two  $d$ -simplices—if it is not on the boundary of  $[-1, n + 1]^d$ .

<sup>1</sup>Using “he” here is not politically correct.

With this the (constructive) proof that we gave before for the 2-dimensional HEX theorem generalizes to the following: we start at the  $d$ -simplex


$$\Delta_0 := \text{conv}\{-\mathbf{1}, -\mathbf{1} + \mathbf{e}_1, -\mathbf{1} + \mathbf{e}_1 + \mathbf{e}_2, \dots, -\mathbf{e}_{d-1} - \mathbf{e}_d, -\mathbf{e}_d, \mathbf{0}\}$$

for which the facet  $((d-1)\text{-face}) \text{conv}\{-\mathbf{1}, -\mathbf{1} + \mathbf{e}_1, \dots, -\mathbf{e}_{d-1} - \mathbf{e}_d, -\mathbf{e}_d\}$  is completely colored. This simplex is shaded in the following figure for  $H(5, 2)$ , which depicts the same final position that we considered before.



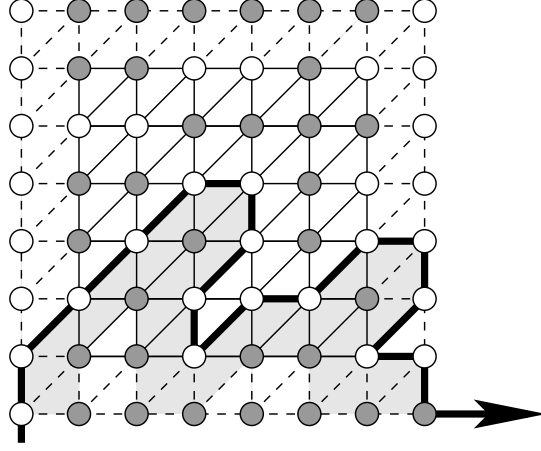
Now we construct a sequence of completely colored  $d$ -dimensional simplices that starts at  $\Delta_0$ : we find the second completely colored  $(d-1)$ -face of  $\Delta_0$ , find the second completely colored  $d$ -simplex it is contained in, etc. Thus we find a chain of completely colored  $d$ -simplices that ends on the boundary of  $[-1, n+1]^d$ —at a different simplex than the one we started from. In particular, the last  $d$ -simplex in the chain has a completely colored facet (a  $(d-1)$ -face) in the boundary, and by construction this facet has to lie in a hyperplane  $H_i^+ = \{\mathbf{x} : x_i = n+1\}$ . (In fact, at this point we check that every completely colored  $(d-1)$ -simplex in the boundary of  $H(n, d)$  is contained in one of the hyperplanes  $H_i^+$ , with the sole exception of the boundary facet of our starting  $d$ -simplex.) And the chain of  $d$ -simplices thus provides us with an  $i$ -colored path from the  $i$ -colored vertex

$$-\mathbf{1} + \mathbf{e}_1 + \dots + \mathbf{e}_{i-1} \in H_i^- = \{\mathbf{x} : x_i = -1\}$$

to the  $i$ -colored vertex in  $H_i^+$ : so the  $i$ -th player wins. 

Our drawing illustrates the chain of completely colored simplices (shaded)

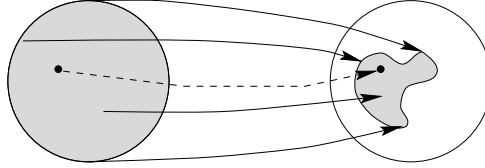
and the sequence of (white) vertices for the winning path that we get from it.



Now we will proceed from the discrete mathematics setting of the HEX game to the continuous world of topological fixed point theorems. Here are three versions of the Brouwer fixed point theorem.

**Theorem 1.4** (Brouwer fixed point theorem). *The following are equivalent (and true):*

- (Br1) *Every continuous map  $f: B^d \rightarrow B^d$  has a fixed point.*
- (Br2) *Every continuous map  $f: B^d \rightarrow S^{d-1}$  has a fixed point.*
- (Br3) *Every null-homotopic map  $f: S^{d-1} \rightarrow S^{d-1}$  has a fixed point.*

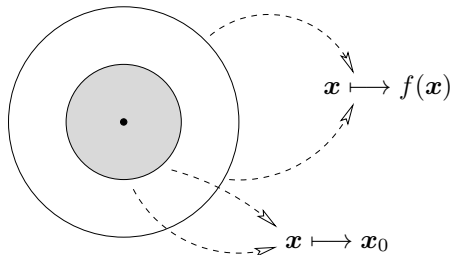


(The term *null-homotopic* that appears here refers to a map that can be deformed to a constant map; see the proof below.)

**Proof of the equivalences.** (Br1) $\implies$ (Br2) is trivial, since  $S^{d-1} \subseteq B^d$ .


For (Br2) $\implies$ (Br3) let  $h: S^{d-1} \times [0, 1] \rightarrow S^{d-1}$  be a null-homotopy for  $f$ , i. e., a continuous map that interpolates between our original map  $f$  and a constant map, with  $h(x, 0) = f(x)$  and  $h(x, 1) = x_0$  for all  $x \in S^{d-1}$ . From this we construct a continuous map  $F: B^d \rightarrow S^{d-1}$  that extends  $f$ , by

$$F(x) := \begin{cases} h(\frac{x}{|x|}, 2 - 2|x|) & \text{if } \frac{1}{2} \leq |x| \leq 1, \\ x_0 & \text{for } |x| \leq \frac{1}{2}. \end{cases}$$



This map is continuous, and by (Br2) it has a fixed point, which must lie in the image, that is, in  $S^{d-1}$ .

For the converse, (Br3) $\implies$ (Br2), let  $f: B^d \rightarrow S^{d-1}$  be continuous. Then the restriction  $f|_{S^{d-1}}$  is null-homotopic, since  $h(\mathbf{x}; t) := f((1-t)\mathbf{x})$  provides a null-homotopy. Thus by (Br3)  $f|_{S^{d-1}}$  has a fixed point, hence so does  $f$ .

Finally, we get (Br2) $\implies$ (Br1): if  $f: B^d \rightarrow B^d$  has no fixed point, then we set  $g(\mathbf{x}) := \frac{f(\mathbf{x}) - \mathbf{x}}{|f(\mathbf{x}) - \mathbf{x}|}$ . This defines a map  $g: B^d \rightarrow S^{d-1}$  that has a fixed point  $\mathbf{x}_0 \in S^{d-1}$  by (Br2), with  $\mathbf{x}_0 = \frac{f(\mathbf{x}_0) - \mathbf{x}_0}{|f(\mathbf{x}_0) - \mathbf{x}_0|}$ . But this implies  $f(\mathbf{x}_0) = \mathbf{x}_0(1+t)$  for  $t := |f(\mathbf{x}_0) - \mathbf{x}_0| > 0$ , and this is impossible for  $\mathbf{x}_0 \in S^{d-1}$ . 

In the following we use the unit cube  $[0, 1]^d$  instead of the ball  $B^d$ : it should be clear that the Brouwer fixed point theorem equally applies to self-maps of any domain  $D$  that is homeomorphic to  $B^d$ , resp. of the boundary  $\partial D$  of such a domain.

**Proof of the Brouwer fixed point theorem** (“HEX  $\implies$  (Br1)”). If  $f: [0, 1]^d \rightarrow [0, 1]^d$  has no fixed point, then for some  $\varepsilon > 0$  we have that  $|f(\mathbf{x}) - \mathbf{x}|_\infty \geq \varepsilon$  for all  $\mathbf{x} \in [0, 1]^d$  (namely, one can take  $\varepsilon := \min\{|f(\mathbf{x}) - \mathbf{x}|_\infty : \mathbf{x} \in [0, 1]^d\}$ , which exists since  $[0, 1]^d$  is compact).

Furthermore, any continuous function on the compact set  $[0, 1]^d$  is uniformly continuous (see e.g. Munkres [Mun00, §27]), hence there exists some  $\delta > 0$  such that  $|\mathbf{x} - \mathbf{x}'|_\infty < \delta$  implies  $|f(\mathbf{x}) - f(\mathbf{x}')|_\infty < \varepsilon$ . We take  $\delta < \varepsilon$  (without loss of generality), and then choose  $n$  with  $\frac{1}{n} < \delta$ .

From  $f$ , we now define a coloring of  $H(n, d)$ , by setting

$$\kappa(\mathbf{v}) := \min\{i : |f_i(\frac{\mathbf{v}}{n}) - \frac{v_i}{n}| \geq \varepsilon\}$$

for the interior vertices  $\mathbf{v} \in H(n, d)$ , where  $f_i$  denotes the  $i$ th component of  $f$ . This is well-defined, since  $\frac{\mathbf{v}}{n} \in [0, 1]^d$ , and thus at least one component of  $f(\frac{\mathbf{v}}{n}) - \frac{\mathbf{v}}{n}$  has to be at least  $\varepsilon$  in its absolute value. Now the  $d$ -dimensional HEX theorem guarantees, for some  $i$ , a chain  $\mathbf{v}^0, \mathbf{v}^1, \dots, \mathbf{v}^N$  of vertices of color  $i$ , where  $v_i^0 = 0$  and  $v_i^N = n$ . Furthermore, we know  $|f_i(\frac{\mathbf{v}^k}{n}) - \frac{v_i^k}{n}| \geq \varepsilon$  for  $0 \leq k \leq N$ . And at the ends we of the chain know the signs:

$$f(\frac{\mathbf{v}^0}{n}) \in [0, 1]^d \text{ implies } f_i(\frac{\mathbf{v}^0}{n}) \geq 0 \text{ and hence } f_i(\frac{\mathbf{v}^0}{n}) - \frac{v_i^0}{n} \geq \varepsilon, \text{ and}$$

$$f(\frac{\mathbf{v}^N}{n}) \in [0, 1]^d \text{ implies } f_i(\frac{\mathbf{v}^N}{n}) \leq 1 \text{ and hence } f_i(\frac{\mathbf{v}^N}{n}) - \frac{v_i^N}{n} \leq -\varepsilon.$$

Hence, for some  $k \in \{1, 2, \dots, N\}$  we must have a sign change:

$$f_i(\frac{\mathbf{v}^{k-1}}{n}) - \frac{v_i^{k-1}}{n} \geq \varepsilon \text{ and } f_i(\frac{\mathbf{v}^k}{n}) - \frac{v_i^k}{n} \leq -\varepsilon.$$

All this taken together provides a contradiction, since

$$|\frac{v_i^{k-1}}{n} - \frac{v_i^k}{n}|_\infty = \frac{1}{n} < \delta$$

whereas

$$|f(\frac{\mathbf{v}^{k-1}}{n}) - f(\frac{\mathbf{v}^k}{n})|_\infty \geq |f_i(\frac{\mathbf{v}^{k-1}}{n}) - f_i(\frac{\mathbf{v}^k}{n})| \geq 2\varepsilon - |\frac{v_i^{k-1}}{n} - \frac{v_i^k}{n}| \geq 2\varepsilon - \frac{1}{n} > 2\varepsilon - \delta > \varepsilon.$$






**Proof that the Brouwer fixed point theorem implies the HEX theorem** (“Br1  $\implies$  HEX”). Assume we have a coloring of  $H(n, d)$ . We use it to define a map  $[0, n]^d \rightarrow [0, n]^d$ , as follows: on the points in  $\{0, 1, \dots, n\}^d$  we define

$$f(\mathbf{v}) = \begin{cases} \mathbf{v} + \mathbf{e}_i & \text{if } \mathbf{v} \text{ has color } i, \text{ and there is a path on vertices of color } i \\ & \text{that connects } \mathbf{v} \text{ to a vertex } \mathbf{w} \text{ with } w_i = 0 \\ \mathbf{v} - \mathbf{e}_i & \text{if } \mathbf{v} \text{ has color } i, \text{ but there is no such path.} \end{cases}$$

If for the given coloring there is no winning path for HEX, then these definitions do not map any point  $\mathbf{v}$  outside  $[0, n]^d$ . Hence this defines a simplicial map  $f: [0, n]^d \rightarrow [0, n]^d$ , by linear extension on the simplices of the triangulation  $\Delta(n, d)$  that we have considered before.

The following two observations now give us a contradiction, showing that this  $f$  cannot have a fixed point:

- If  $\Delta = \text{conv}\{\mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^d\} \subseteq \mathbb{R}^d$  is a simplex and  $f: \Delta \rightarrow \mathbb{R}^d$  is a linear map defined by  $f(\mathbf{v}^i) = \mathbf{v}^i + \mathbf{w}^i$ , then  $f$  has a fixed point on  $\Delta$  if and only if  $\mathbf{0} \in \text{conv}\{\mathbf{w}^0, \dots, \mathbf{w}^d\}$ .
- If  $\mathbf{v}, \mathbf{v}'$  are adjacent vertices, then we cannot get  $f(\mathbf{v}) = \mathbf{v} - \mathbf{e}_i$  and  $f(\mathbf{v}') = \mathbf{v}' + \mathbf{e}_i$ . Hence for each simplex of  $\Delta(n, d)$ , all the vectors  $\mathbf{w}^i$  lie in one orthant of  $\mathbb{R}^d$ ! 

## Exercises

1. In the proof of the Brouwer fixed point theorem (Thm. 1.4, (Br2) $\implies$ (Br3)), we could simply have put  $F(\mathbf{x}) := h(\frac{\mathbf{x}}{|\mathbf{x}|}, 1 - |\mathbf{x}|)$ . Is this continuous?

## 2 Who wins HEX?

So, who can win the 2-dimensional HEX game? A simple but ingenious argument due to John Nash, known as “stealing a strategy,” shows that on a square board the first player (“White”) always has a winning strategy. In the following we first define winning strategies, then find that one of the players has one, and finally conclude that the first player has one. Still: the proof will be non-constructive, and we don’t know how to win HEX. So, the game still remains interesting ...

**Definition 2.1.** A *strategy* is a set of rules that tells a player which move to make (i. e., which tile to color) for every legal position on the board. A *winning strategy* here guarantees to lead to a win, starting from an empty board, for all possible moves of the opponent.

A *position* of the HEX game is a board on which some tiles may have been colored white or black, together with the information who moves next (unless all tiles are colored). A position is *legal* if it can occur in a HEX game: that is, if either White moves next, and the numbers of white and black tiles agree, or if Black moves next, and White has one more tile.

A *winning position for White* is a position such that White has a *winning strategy* that tells him how to proceed (for arbitrary moves of Black) and guarantees a win. Similarly, a *winning position for Black* has a *winning strategy* that guarantees to lead Black to a win.

**Lemma 2.2.** *Every (legal) position for HEX is either a winning position for White or a winning position for Black.*

**Proof.** Here we proceed by induction on the number  $g$  of “grey” tiles (i. e., “free” positions on the board). If no grey tiles are present ( $g = 0$ ), then one of the players has won—by the HEX theorem.

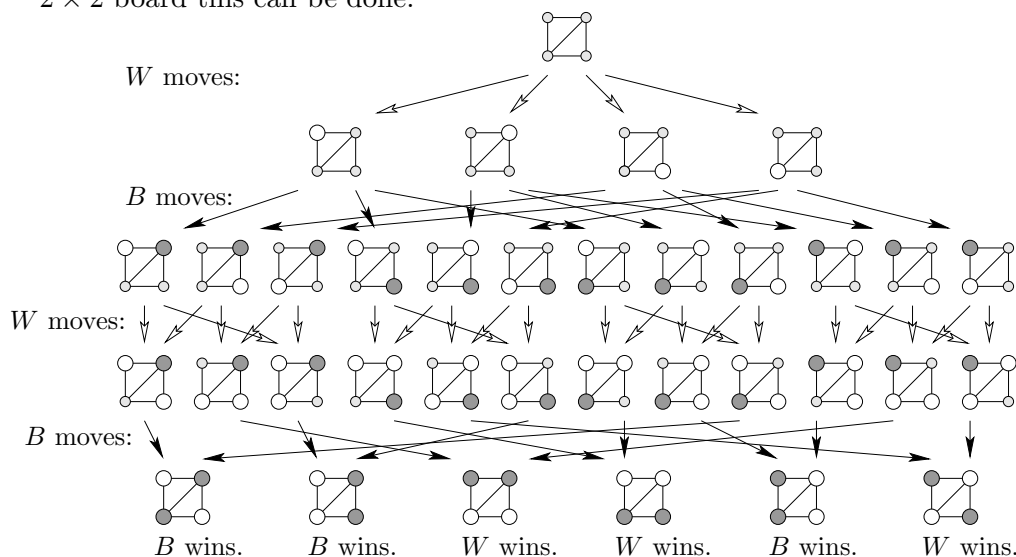
If  $g > 0$  and White is to move, then any move that White could make reduces  $g$ , and thus (by induction) produces a winning position for one of the players. If there is a move that leads to a winning position for White, then this is really nice and great for White: this makes the present position into a winning position for White, and any such move can be used for a winning position for White. Otherwise—too bad: if every possible move for White produces a winning position for Black, then we are at a winning position for Black already.

And the same argument applies for  $g > 0$  if Black is to move. 

Of course, the argument given here is *much* more general: essentially we have proved that for any finite deterministic 2-person game without a draw and with “complete information” there is a winning strategy for one of the players. (This is a theorem of Zermelo, which was rediscovered by von Neumann and Morgenstern). Furthermore, for games where a draw is possible either one player has a winning strategy, or *both* players can force a draw. We refer to Exercise 7, and to Blackwell & Girshick [BG54, p. 21].

For HEX, Lemma 2.2 shows that at the beginning (for the starting position, where all tiles are grey, and White is to move), there is a winning strategy either for White or for Black. But who is the winner?


Our first attempt might be to follow the proof of Lemma 2.2. Only for the  $2 \times 2$  board this can be done:



In this drawing, you can decide for every position whether it is a winning position for White or for Black, starting with the bottom row ( $g = 0$ ) that has three winning positions for each player, ending at the top node ( $g = 4$ ), which turns out to be a winning position for White.

For larger boards, this approach is hopeless—after all, there are  $\binom{n^2}{\lfloor n^2/2 \rfloor}$  final positions to classify for “ $g = 0$ ,” and from this one would have to work one’s way up to the top node of a huge tree (of height  $n^2$ ). Nevertheless, people have worked out winning strategies for White on the  $n \times n$  boards for  $n \leq 5$  (see Gardner [Gar58]).

**Theorem 2.3.** *For the HEX game played on a HEX board with equal side lengths, White (the first player) has a winning strategy.*

**Proof.** Assume not: then by Lemma 2.2 Black has a winning strategy. But then White can start with an arbitrary move, and then—using the symmetry of the board and of the rules—just ignore his first tile, and follow Black’s winning strategy “for the second player.” This strategy will tell White always which move to take. Here the “extra” white tiles cannot hurt White: if the move for White asks to occupy a tile that is already white, then an arbitrary move is o.k. for White. But this “stealing a strategy argument” produces a winning strategy for White, contradicting our assumption! 

## Notes

Gale’s beautiful paper [Gal79] was the source and inspiration for our treatment of Brouwer’s fixed point theorem in terms of the HEX game. Nash’s analysis for the winning strategies for HEX is from Gardner’s classical account in [Gar58], some of which reappears in Milnor’s [Mil95]. See also the accounts in Jensen & Toft [JT95, Sect. 17.14], and in Berlekamp, Conway & Guy [BCG82, p. 680],

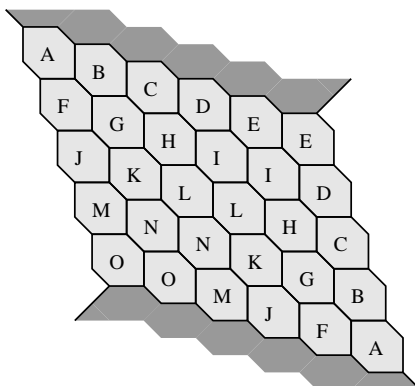
where other cases of “strategy stealing” are discussed. (A theoretical set-up for this is in Hales & Jewett [HJ63, Sect. 3].)

The traditional combinatorial approach to the Brouwer fixed point theorem is via Sperner’s lemma [Spe28]; see e.g. the presentation in [AZ14]. A more geometric version of the combinatorial lemmas is given by Mani [Man67].

## Exercises

1. Stir your coffee cup. Show that the (moving, but flat) surface has at every moment at least one point that stands still (has velocity zero).
2. Prove that if you tear a sheet of paper from your notebook, crumble it into a small ball, and put that down on your notebook, then at least one point of the sheet comes to rest exactly on top of its original position. Could it happen that there are exactly two such points?
3. For HEX on a  $3 \times 3$  board, how large is the tree of possible positions?
4. Can you write a computer program that plays HEX and wins (sometimes) [Bro00]?
5. For  $d$ -dimensional HEX, is there always some “short” winning path? Show that for every  $d \geq 2$  there is a constant  $c_d$  such that for all  $n$  there is a final configuration such that only one player wins, but his shortest path uses more than  $c_d \cdot n^d$  tiles.
6. Construct an algorithm that, for given  $\varepsilon > 0$  and  $f: [0, 1]^2 \rightarrow [0, 1]^2$ , calculates a point  $x_0 \in [0, 1]^2$  with  $|f(x) - x| < \varepsilon$ . [Gal79, p. 827]
7. If in a complete information two player game a draw is possible, argue why either one of the players has a winning strategy, or *both* can force at least a draw.
8. Prove that for 2-dimensional HEX, not both players can win! For this, prove and use the “polygonal Jordan curve theorem”: any simple closed polygon in the plane uniquely divides the plane into “inside” and “outside.”  
(The general Jordan curve theorem for simple “Jordan arcs” in the plane has extensive discussions in many books; see for example Munkres [Mun00], Stillwell [Sti93, Sect. 0.3], or Thomassen [Tho92].)
9. On an  $(m \times n)$ -board that is not square (that is,  $m \neq n$ ), the player who gets the longer sides, and hence the shorter distance to bridge by a winning path, has a winning strategy. (Our figure illustrates the case of a  $(6 \times 5)$ -board, where the claim is that Black has a winning strategy.)
  - (i) Show that for this, it is sufficient to consider the case where  $m = n + 1$  (i. e., the second player Black, who gets the longer side, has a sure

win).

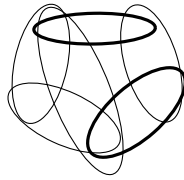


- (ii) Show that in the situation of (i), Black has the following winning strategy. Label the tiles in the “symmetric” way that is indicated by the figure, such there are two tiles of each label. The strategy for Black is to always take the second tile that has the same label as the one taken by White. Why will this strategy always win for Black? (Hint: you will need the Jordan curve theorem.) (This is in Gardner [Gar58] and in Milnor [Mil95], but neither source gives the proof. You’ll have to work yourself!)

### 3 Piercing multiple intervals

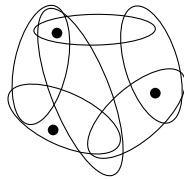
**Packing number and transversal number.** Let  $\mathcal{S}$  be a system of sets on a ground set  $X$ ; both  $\mathcal{S}$  and  $X$  may generally be infinite. The *packing number* of  $\mathcal{S}$ , usually denoted by  $\nu(\mathcal{S})$  and often also called the *matching number*, is the maximum cardinality of a system of pairwise disjoint sets in  $\mathcal{S}$ :

$$\nu(\mathcal{S}) = \sup\{|\mathcal{M}| : \mathcal{M} \subseteq \mathcal{S}, M_1 \cap M_2 = \emptyset \text{ for all } M_1, M_2 \in \mathcal{M}, M_1 \neq M_2\}.$$



The *transversal number* or *piercing number* of  $\mathcal{S}$  is the smallest number of points of  $X$  that capture all the sets in  $\mathcal{S}$ :

$$\tau(\mathcal{S}) = \min\{|T| : T \subseteq X, S \cap T \neq \emptyset \text{ for all } S \in \mathcal{S}\}.$$



A subsystem  $\mathcal{M} \subseteq \mathcal{S}$  of pairwise disjoint sets is usually called a *matching* (this refers to the graph-theoretical matching, which is a system of pairwise disjoint edges), and a set  $T \subseteq X$  intersecting all sets of  $\mathcal{S}$  is referred to as a *transversal* of  $\mathcal{S}$ . Clearly, any transversal is at least as large as any matching, and so always

$$\nu(\mathcal{S}) \leq \tau(\mathcal{S}).$$

In the reverse direction, very little can be said in general, since  $\tau(\mathcal{S})$  can be arbitrarily large even if  $\nu(\mathcal{S}) = 1$ . As a simple geometric example, we can take the plane as the ground set of  $\mathcal{S}$  and let the sets of  $\mathcal{S}$  be lines in general position. Then  $\nu = 1$ , since every two lines intersect, but  $\tau \geq \frac{1}{2}|\mathcal{S}|$ , because no point is contained in more than two of the lines.

One of the basic general questions in combinatorics asks for interesting special classes of set systems where the transversal number can be bounded in terms of the matching number.<sup>2</sup> Many such examples come from geometry.

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<sup>2</sup>This kind of problem is certainly not restricted to combinatorics. For example, if  $\mathcal{S}$  is the system of all open sets in a topological space,  $\tau(\mathcal{S})$  is the minimum size of a dense set and is called the *density*, while  $\nu(\mathcal{S})$  is known as the *Souslin number* or *cellularity* of the space. In 1920, Souslin asked whether a linearly ordered topological space exists (the open sets are unions of open intervals) with countable  $\nu$  but uncountable  $\tau$ . It turned out in the 1970s that the answer depends on the axioms one is willing to assume beyond the usual (ZFC) axioms of set theory. For example, it is yes if one assumes the continuum hypothesis; see e. g. [Eng77].

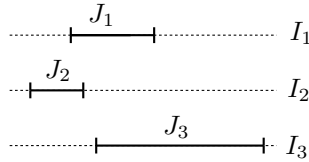
Here we restrict our attention to one particular type of systems, the *d-intervals*, where the best results have been obtained by topological methods.

**Fractional packing and transversal numbers.** Before introducing *d-intervals*, we mention another important parameter of a set system, which always lies between  $\nu$  and  $\tau$  and often provides useful estimates for  $\nu$  or  $\tau$ . This parameter can be introduced in two seemingly different ways. For simplicity, we restrict ourselves to finite set systems (on possibly infinite ground sets). A *fractional packing* for a finite set system  $\mathcal{S}$  on a ground set  $X$  is a function  $w: \mathcal{S} \rightarrow [0, 1]$  such that for each  $x \in X$ , we have  $\sum_{S \in \mathcal{S}: x \in S} w(S) \leq 1$ . The *size* of a fractional packing  $w$  is  $\sum_{S \in \mathcal{S}} w(S)$ , and the *fractional packing number*  $\nu^*(\mathcal{S})$  is the supremum of the sizes of all fractional packings for  $\mathcal{S}$ . So in a fractional packing, we can take, say, one-third of one set and two-thirds of another, but at each point, the fractions for the sets containing that point must add up to at most 1. We always have  $\nu(\mathcal{S}) \leq \nu^*(\mathcal{S})$ , since a packing  $\mathcal{M}$  defines a fractional packing  $w$  by setting  $w(S) = 1$  for  $S \in \mathcal{M}$  and  $w(S) = 0$  otherwise.

Similar to the fractional packing, one can also introduce a fractional version of a transversal. A *fractional transversal* for a (finite) set system  $\mathcal{S}$  on a ground set  $X$  is a function  $\varphi: X \rightarrow [0, 1]$  attaining only finitely many nonzero values such that for each  $S \in \mathcal{S}$ , we have  $\sum_{x \in S} \varphi(x) \geq 1$ . The size of a fractional transversal  $\varphi$  is  $\sum_{x \in X} \varphi(x)$ , and the *fractional transversal number*  $\tau^*(\mathcal{S})$  is the infimum of the sizes of fractional transversals.

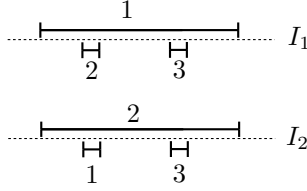
By the duality of linear programming (or by the theorem about separation of disjoint convex sets by a hyperplane), it follows that  $\nu^*(\mathcal{S}) = \tau^*(\mathcal{S})$  for any finite set system  $\mathcal{S}$ . When trying to bound  $\tau$  in terms of  $\nu$ , in many instances it proved very useful to bound  $\nu^*$  as a function of  $\nu$  first, and then  $\tau$  in terms of  $\tau^*$ . The proof presented below follows a somewhat similar approach.

**The *d-intervals*.** Let  $I_1, I_2, \dots, I_d$  be disjoint parallel segments of unit length in the plane. A set  $J \subset \bigcup_{i=1}^d I_i$  is a *d-interval* if it intersects each  $I_i$  in a closed interval. We denote this intersection by  $J_i$  and call it the *i*th component of  $J$ . The drawing shows a 3-interval:



Intersection and piercing for *d-intervals* are taken in the set-theoretical sense, i.e. two *d-intervals* intersect if, for some  $i$ , their *i*th components intersect, etc.

The 1-intervals, which are just intervals in the usual sense, behave nicely with respect to packing and piercing: for any family  $\mathcal{F}$  of intervals, we have  $\nu(\mathcal{F}) = \tau(\mathcal{F})$  (this is well-known and easy to prove). The following family  $\mathcal{F}$  of three 2-intervals



has  $\nu(\mathcal{F}) = 1$  while  $\tau(\mathcal{F}) = 2$ . By taking multiple copies of this family, one obtains families with  $\tau = 2\nu$  for all values of  $\nu$ .

Gyárfás and Lehel [GL70] showed by elementary methods that for any  $d$  and any family  $\mathcal{F}$  of  $d$ -intervals,  $\tau(\mathcal{F})$  can be bounded by a function of  $\nu(\mathcal{F})$  (also see [GL85]). Their function was rather large (about  $\nu^{d!}$  for  $d$  fixed). After an initial breakthrough by Tardos [Tar95], who proved  $\tau(\mathcal{F}) \leq 2\nu(\mathcal{F})$  for any family of 2-intervals, Kaiser [Kai97] obtained the following result:

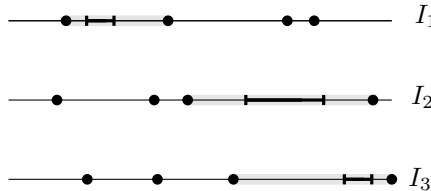
**Theorem 3.1** (The Tardos–Kaiser theorem on  $d$ -intervals). *Every family  $\mathcal{F}$  of  $d$ -intervals,  $d \geq 2$ , has a transversal of size at most  $(d^2 - d) \cdot \nu(\mathcal{F})$ .*

Here we present a proof using Brouwer’s fixed point theorem. Alon [Alo98] found a short non-topological proof of the slightly weaker bound  $\tau(\mathcal{F}) \leq 2d^2\nu(\mathcal{F})$ .

**Proof.** Let  $\mathcal{F}$  be a fixed system of  $d$ -intervals with  $\nu(\mathcal{F}) = k$ , and let  $t = t(d, k)$  be a suitable (yet undetermined) integer. The general plan of the proof is this: Assuming that there is no transversal of  $\mathcal{F}$  of size  $dt$ , we show by a topological method that the fractional packing number  $\nu^*(\mathcal{F})$  is at least  $t + 1$ . Then a simple combinatorial argument proves that the packing number  $\nu(\mathcal{F})$  is at least  $\frac{t+1}{d}$ , which leads to  $t < d^2 \cdot \nu(\mathcal{F})$ . An sharper combinatorial reasoning in this step leads to the slightly better bound in the theorem.

Our candidates for a transversal of  $\mathcal{F}$  are all sets  $T$  with each  $T_i = T \cap I_i$  having exactly  $t$  points; so  $|T| = td$ . For technical reasons, we also permit that some of the  $t$  points in  $I_i$  coincide, so  $T$  can be a multiset.

The letter  $T$  could also abbreviate a *trap*. The trap is set to catch all the  $d$ -intervals in  $\mathcal{F}$ , but if it is not set well enough, some of the  $d$ -intervals can escape. Each of them escapes through a hole in the trap, namely through a *d-hole*. The points of  $T_i$  cut the segment  $I_i$  into  $t + 1$  open intervals (some of them may be empty), and these are the *holes in  $I_i$* ; they are numbered 1 through  $t + 1$  from left to right. A *d-hole* consists of  $d$  holes, one in each  $I_i$ . The *type* of a *d-hole*  $H$  is the set  $\{(1, j_1), (2, j_2), \dots, (d, j_d)\}$ , where  $j_i \in [t+1]$  is the number of the hole in  $I_i$  contained in  $H$ . A *d-interval*  $J \in \mathcal{F}$  *escapes* through a *d-hole*  $H$  if it is contained in the union of its holes. The drawing shows a 3-hole, of type  $\{(1, 2), (2, 4), (3, 4)\}$ , and a 3-interval escaping through it:



Let  $\mathcal{H}_0$  be the hypergraph with vertex set  $[d] \times [t+1]$  and with edges being all possible types of  $d$ -holes; for example, the hole in the picture yields the edge



$\{(1, 2), (2, 4), (3, 4)\}$ . So  $\mathcal{H}_0$  is a complete  $d$ -partite  $d$ -uniform hypergraph (we will meet such hypergraphs several times in this book). By saying that a  $J \in \mathcal{F}$  escapes through an edge  $H$  of  $\mathcal{H}_0$ , we mean that  $J$  escapes through the  $d$ -hole (uniquely) corresponding to  $H$ .

Next, we define weights on the edges of  $\mathcal{H}_0$ ; these weights depend on the set  $T$  (and also on  $\mathcal{F}$  but this is considered fixed). The weight of an edge  $H$  is

$$q_H = \sup\{\text{dist}(J, T) : J \in \mathcal{F}, J \text{ escapes through } H\}.$$

Here  $\text{dist}(J, T) = \max_i \{\text{dist}(J_i, T_i)\}$  and  $\text{dist}(J_i, T_i)$  is the distance of the  $i$ th component of  $J$  to the closest point of  $T_i$ . Thus,  $q_H$  can be interpreted as the slimmest margin by which a  $d$ -interval  $J$  escaping through  $H$  avoids being trapped. If no members of  $\mathcal{F}$  escape through  $H$ , we define  $q_H$  as 0. Note that this is the only case where  $q_H = 0$ ; otherwise, if anything escapes, it does so by a positive margin, since we are dealing with closed intervals.

From the edge weights, we derive weights of vertices: the weight  $w_v$  of a vertex  $v = (i, j)$  is the sum of the weights of the edges of  $\mathcal{H}_0$  containing  $v$ . These weights, too, are functions of  $T$ ; to emphasize this, we can write  $w_v = w_v(T)$ .

**Lemma 3.2.** *For any  $d \geq 1$ ,  $t \geq 1$ , and any  $\mathcal{F}$ , there is a choice of  $T$  such that all the vertex weights  $w_v(T)$ ,  $v \in [d] \times [t+1]$ , coincide.*

It is this lemma whose proof is topological. We postpone that proof and finish the combinatorial part.

Let us suppose that a trap  $T$  was chosen as in the lemma, with  $w_v(T) = W$  for all  $v$ . If  $W = 0$  then  $T$  is a transversal, since all edge weights are 0 and no  $J \in \mathcal{F}$  escapes. So suppose that  $W > 0$ .

Let  $\mathcal{H} = \mathcal{H}(T) \subseteq \mathcal{H}_0$ , the *escape hypergraph* of  $T$ , consist of the edges of  $\mathcal{H}_0$  with nonzero weights. Note that

$$\nu(\mathcal{H}) \leq \nu(\mathcal{F}). \quad (1)$$

Indeed, given a matching  $\mathcal{M}$  in  $\mathcal{H}$ , for each edge  $H \in \mathcal{M}$  choose a  $J \in \mathcal{F}$  escaping through  $H$ —this gives a matching in  $\mathcal{F}$ .

We note that the re-normalized edge weights  $\tilde{q}_H = \frac{1}{W} q_H$  determine a fractional packing in  $\mathcal{H}$  (since the weights at each vertex sum up to 1). For the size of this fractional packing, which is the total weight of all vertices, we find by double counting


$$\nu^*(\mathcal{H}) \geq \sum_{H \in \mathcal{H}} \tilde{q}_H = \frac{1}{d} \sum_{H \in \mathcal{H}} \sum_{v \in H} \tilde{q}_H = \frac{1}{d} \sum_{v \in [d] \times [t+1]} \frac{w_v}{W} = \frac{1}{d} \sum_v 1 = t + 1.$$

The last step is to show that  $\nu(\mathcal{H})$  cannot be small if  $\nu^*(\mathcal{H})$  is large. Here is a simple argument leading to a slightly suboptimal bound, namely  $\nu(\mathcal{H}) \geq \frac{1}{d} \nu^*(\mathcal{H})$ .

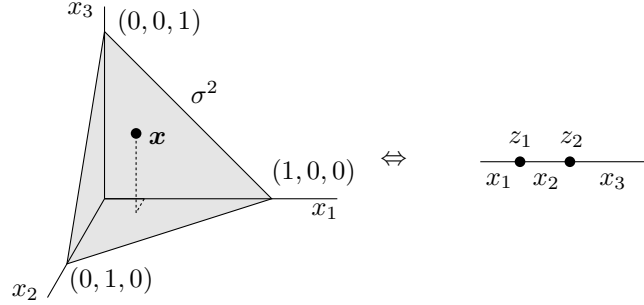
Given a fractional matching  $\tilde{q}$  of size  $t + 1$  in  $\mathcal{H}$ , a matching can be obtained by the following greedy procedure: Pick an edge  $H_1$  and discard all edges intersecting it, pick  $H_2$  among the remaining edges, etc., until all edges are exhausted. The  $\tilde{q}$ -weight of  $H_i$  plus all the edges discarded with it is at most

$d = |H_i|$ , while all edges together have weight  $t + 1$ . Thus, the number of steps, and also the size of the matching  $\{H_1, H_2, \dots\}$ , is at least  $\lceil \frac{t+1}{d} \rceil$ .

If we set  $t = d \cdot \nu(\mathcal{F})$ , we get  $\nu(\mathcal{H}) > \nu(\mathcal{F})$ , which contradicts (1). Therefore, for this choice of  $t$ , all the vertex weights must be 0 and  $T$  as in Lemma 3.2 is a transversal of  $\mathcal{F}$  of size at most  $d^2 \nu(\mathcal{F})$ .

The improved bound  $\tau(\mathcal{F}) \leq (d^2 - d) \cdot \nu(\mathcal{F})$  for  $d \geq 3$  follows similarly using a theorem of Füredi [Für81], which implies that the matching number of any  $d$ -uniform  $d$ -partite hypergraph  $\mathcal{H}$  satisfies  $\nu(\mathcal{H}) \leq (d - 1)\nu^*(\mathcal{H})$ . (For  $d = 2$ , a separate argument needs to be used, based on a theorem of Lovász stating that  $\nu^*(G) \leq \frac{3}{2}\nu(G)$  for all graphs  $G$ .) The Tardos–Kaiser theorem 3.1 is proved. 

**Proof of Lemma 3.2.** Let  $\sigma^t$  denote the standard  $t$ -dimensional simplex in  $\mathbb{R}^{t+1}$ , i.e. the set  $\{\mathbf{x} \in \mathbb{R}^{t+1} : x_j \geq 0, x_1 + \dots + x_{t+1} = 1\}$ . A point  $\mathbf{x} \in \sigma^t$  defines a  $t$ -point multiset  $\{z_1, z_2, \dots, z_t\} \subset [0, 1]$ ,  $z_1 \leq z_2 \leq \dots \leq z_t$ , by setting  $z_k = \sum_{j=1}^k x_j$ . Here is a picture for  $t = 2$ :



A candidate transversal  $T$  with  $t$  points in each  $I_i$  can thus be defined by an ordered  $d$ -tuple  $(\mathbf{x}_1, \dots, \mathbf{x}_d)$  of points,  $\mathbf{x}_i \in \sigma^t$ , where  $\mathbf{x}_i$  determines  $T_i$ . Such an ordered  $d$ -tuple can be regarded as a single point  $\mathbf{x}$  in the Cartesian product  $P = \sigma^t \times \sigma^t \times \dots \times \sigma^t = (\sigma^t)^d$ . To each  $\mathbf{x} \in P$ , we have thus assigned a candidate transversal  $T(\mathbf{x})$ .

For each vertex  $v = (i, j)$  of the hypergraph  $\mathcal{H}_0$ , we define the function  $g_{ij} : P \rightarrow \mathbb{R}$  by  $g_{ij}(\mathbf{x}) = w_{(i,j)}(T(\mathbf{x}))$ , where  $w_v(T)$  is the vertex weight. This is a continuous function of  $\mathbf{x}$ .

We note that for each  $\mathbf{x}$ , the sum

$$S_i(\mathbf{x}) = \sum_{j=1}^{t+1} g_{ij}(\mathbf{x})$$

is independent of  $i$ ; this is because  $S_i(\mathbf{x})$  equals the sum of the weights of all edges. So we can write just  $S(\mathbf{x})$  instead of  $S_i(\mathbf{x})$ .

If there is an  $\mathbf{x} \in P$  with  $S(\mathbf{x}) = 0$ , then all the vertex weights  $w_{(i,j)}(T(\mathbf{x}))$  are 0 and we are done. Otherwise, we define the normalized functions

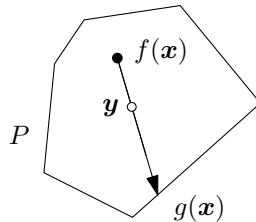
$$f_{ij}(\mathbf{x}) = \frac{1}{S(\mathbf{x})} g_{ij}(\mathbf{x}).$$


For each  $i$ ,  $f_{i1}(\mathbf{x}), \dots, f_{i(t+1)}(\mathbf{x})$  are nonnegative and sum up to 1, and so they are the coordinates of a point in the standard simplex  $\sigma^t$ . All the maps  $f_{ij}$  together can be regarded as a map  $f: P \rightarrow P$ . To prove the lemma, we need to show that the image of  $f$  contains the point of  $P$  with all the  $d(t+1)$  coordinates equal to  $\frac{1}{t+1}$ .

The product  $P$  is a convex polytope, and its nonempty faces are exactly all Cartesian products  $F_1 \times F_2 \times \dots \times F_d$ , where  $F_1, \dots, F_d$  are nonempty faces of  $\sigma^t$  (Exercise 1). We note that for any face  $F$  of  $P$ , we have  $f(F) \subseteq F$ . Indeed, a face  $G$  of  $\sigma^t$  has the form  $G = \{\mathbf{x} \in \sigma^t : x_i = 0 \text{ for all } i \in I\}$ , for some index set  $I$ , and the faces of  $P$  are products of faces  $G$  of this form. So it suffices to know that  $f_{ij}(\mathbf{x}) = 0$  whenever  $(\mathbf{x}_i)_j = 0$ . This holds, since  $(\mathbf{x}_i)_j = 0$  means that the  $j$ th hole in  $I_i$  is empty, so nothing can escape through that hole, and thus  $f_{ij}(\mathbf{x}) = 0$ . The proof of Lemma 3.2 is now reduced to the following statement:

**Lemma 3.3.** *Let  $P$  be a convex polytope and let  $f: P \rightarrow P$  be a continuous mapping satisfying  $f(F) \subseteq F$  for each face<sup>3</sup>  $F$  of  $P$ . Then  $f$  is surjective.*

**Proof.** Since the condition is hereditary for faces, it suffices to show that each point  $\mathbf{y}$  in the interior of  $P$  has a preimage. For contradiction, suppose that some  $\mathbf{y} \in \text{int } P$  is not in the image of  $f$ . For  $\mathbf{x} \in P$ , consider the semiline emanating from  $f(\mathbf{x})$  and passing through  $\mathbf{y}$ , and let  $g(\mathbf{x})$  be the unique intersection of that semiline with the boundary of  $P$ .



This  $g$  is a well-defined and continuous map  $P \rightarrow P$ , and by Brouwer's fixed point theorem, there is an  $\mathbf{x}_0 \in P$  with  $g(\mathbf{x}_0) = \mathbf{x}_0$ . The point  $\mathbf{x}_0$  lies on the boundary of  $P$ , in some proper face  $F$ . But  $f(\mathbf{x}_0)$  cannot lie in  $F$ , because the segment  $\mathbf{x}_0 f(\mathbf{x}_0)$  passes through the point  $\mathbf{y}$  outside  $F$ —a contradiction. 

**Lower bounds.** It turns out that the bound in Theorem 3.1 is not far from being the best possible. In particular, for  $\nu(\mathcal{F}) = 1$  and  $d$  large, the transversal number can be near-quadratic in  $d$ , which is rather surprising. For all  $k$  and  $d$ , systems  $\mathcal{F}$  of  $d$ -intervals can be constructed with  $\nu(\mathcal{F}) = k$  and

$$\tau(\mathcal{F}) \geq c \frac{d^2}{(\log d)^2} k$$

for a suitable constant  $c > 0$  (Matoušek [Mat01]). The construction involves an extension of a construction due to Sgall [Sga96] of certain systems of set

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<sup>3</sup>In fact, it suffices to require  $f(F) \subseteq F$  for each facet of  $P$  (that is, for each face of dimension  $\dim(P) - 1$ ), since each face is the intersection of some facets.

pairs. Here we outline a (non-topological!) proof of a somewhat simpler result concerning families of *homogeneous*  $d$ -intervals, which are unions of at most  $d$  closed intervals on the real line. These are more general than the  $d$ -intervals, but an upper bound only slightly weaker than Theorem 3.1 can be proved for them along the same lines (Exercise 3):  $\tau \leq (d^2 - d + 1)\nu$ .

**Proposition 3.4.** *For every  $d \geq 2$  and  $k \geq 1$ , there exists a system  $\mathcal{F}$  of homogeneous  $d$ -intervals with  $\nu(\mathcal{F}) = k$  and*

$$\tau(\mathcal{F}) \geq c \frac{d^2}{\log d} k.$$

**Proof.** Given  $d$  and  $k$ , we want to construct a system  $\mathcal{F}$  of homogeneous  $d$ -intervals. Clearly, it suffices to consider the case  $k = 1$ , since for larger  $k$ , we can take  $k$  disjoint copies of the  $\mathcal{F}$  constructed for  $k = 1$ . Thus, we want an  $\mathcal{F}$  in which every two  $d$ -intervals intersect and with  $\tau(\mathcal{F})$  large.

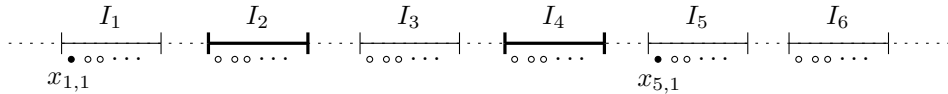
In the construction, we will use homogeneous  $d$ -intervals of a quite special form: each component is either a single point or a unit-length interval. First, it is instructive to see why we cannot get a good example if all the components are only points. In that case, the family  $\mathcal{F}$  is simply a  $d$ -uniform hypergraph (whose vertices happen to be points of the real line). We require that any two edges intersect, and thus any edge is a transversal and we have  $\tau(\mathcal{F}) \leq d$ .

For the actual construction, let  $n$  and  $N$  be integer parameters (whose value will be set later). Let  $V = [n]$  be an index set, and  $I_v$ , for  $v \in V$ , be auxiliary pairwise disjoint unit intervals on the real line. In each  $I_v$ , we choose  $N$  distinct points  $x_{v,i}$ ,  $i = 1, 2, \dots, N$ .

The constructed system  $\mathcal{F}$  consists of homogeneous  $d$ -intervals  $J^1, J^2, \dots, J^N$ . For each  $i = 1, 2, \dots, N$ , we choose auxiliary sets  $B_i \subseteq A_i \subseteq V$ , and we construct  $J^i$  as follows:

$$J^i = \left( \bigcup_{v \in B_i} I_v \right) \cup \{x_{u,i} : u \in A_i \setminus B_i\}.$$

The picture shows an example of  $J^1$  for  $n = 6$ ,  $A_1 = \{1, 2, 4, 5\}$  and  $B_1 = \{2, 4\}$ :



The heart of the proof is the construction of suitable sets  $A_i$  and  $B_i$  on the ground set  $V$ . Since the  $J^i$  should be homogeneous  $d$ -intervals, we obviously require

(C1) For all  $i = 1, 2, \dots, N$ ,  $\emptyset \subset B_i \subseteq A_i$  and  $|A_i| \leq d$ .

The condition that every two members of  $\mathcal{F}$  intersect is implied by the following:


(C2) For all  $i_1, i_2$ ,  $1 \leq i_1 < i_2 \leq N$ , we have  $A_{i_1} \cap B_{i_2} \neq \emptyset$  or  $A_{i_2} \cap B_{i_1} \neq \emptyset$  (or both).

Finally, we want  $\mathcal{F}$  to have no small transversal. Since no two  $d$ -intervals of  $\mathcal{F}$  have a point component in common, a transversal of size  $t$  intersects no more than  $t$  members of  $\mathcal{F}$  in their point components, and all the other members of  $\mathcal{F}$  must be intersected in their interval components. Therefore, the transversal condition translates to

- (C3) Put  $t = cd^2/\log d$  for a sufficiently small constant  $c > 0$ , and let  $\mathcal{B} = \{B_1, B_2, \dots, B_N\}$ . Then  $\tau(\mathcal{B}) \geq 2t$ , and consequently  $\tau(\mathcal{B}') \geq t$  for any  $\mathcal{B}'$  arising from  $\mathcal{B}$  by removing at most  $t$  sets.

A construction of sets  $A_1, \dots, A_N$  and  $B_1, \dots, B_N$  as above was provided by Sgall [Sga96]. His results give the following:


**Proposition 3.5.** *Let  $b$  be a given integer, let  $n \leq cb^2/\log b$  for a sufficiently small constant  $c > 0$ , and let  $B_1, B_2, \dots, B_N$  be  $b$ -element subsets of  $V = [n]$ . Then there exist sets  $A_1, A_2, \dots, A_N$ , with  $B_i \subseteq A_i$ ,  $|A_i| \leq 3b$ , and such that (C2) is satisfied.*

With this proposition, the proof of Proposition 3.4 is easily finished. We set  $b = \lfloor \frac{d}{3} \rfloor$ ,  $n = cb^2/\log b$ , and we let  $B_1, B_2, \dots, B_N$  be all the  $N = \binom{n}{b}$  subsets of  $V$  of size  $b$ . We have  $\tau(\{B_1, \dots, B_N\}) = n - b + 1$  and condition (C3) holds. It remains to construct the sets  $A_i$  according to Proposition 3.5; then (C1) and (C2) are satisfied too. The proof of Proposition 3.4 is concluded by passing from the  $A_i$  and  $B_i$  to the system  $\mathcal{F}$  of homogeneous  $d$ -intervals as was described above. 

**Sketch of proof of Proposition 3.5.** Let  $G = (V, E)$  be a graph on  $n$  vertices of maximum degree  $b$  with the following expander-type property: for any two disjoint  $b$ -element subsets  $A, B \subseteq V$ , there is at least one edge  $e \in E$  connecting a vertex of  $A$  to a vertex of  $B$ . (The existence of such a graph can be easily shown by the probabilistic method; the constant  $c$  arises in this argument. See [Sga96] for references.)

For each  $i$ , let  $v_i$  be an (arbitrary) element of the set  $B_i$ , and let

$$A_i = B_i \cup N(v_i) \cup \left( V \setminus \bigcup_{u \in B_i} N(u) \right),$$

where  $N(v)$  denotes the set of neighbors in  $G$  of a vertex  $v \in V$ . It is easy to check that  $|A_i| \leq 3b$ , and some thought reveals that the condition (C2) is satisfied. 

**A Helly-type problem for  $d$ -intervals.** Kaiser and Rabinovich [KR99] investigated conditions on a family  $\mathcal{F}$  of  $d$ -intervals guaranteeing that  $\mathcal{F}$  can be pierced by a “multipoint,” i.e.  $\tau(\mathcal{F}) = d$  and there is a transversal using one point of each  $I_i$ . They proved the following:

**Theorem 3.6.** *Let  $k = \lceil \log_2(d+2) \rceil$  and let  $\mathcal{F}$  be a family of  $d$ -intervals such that any  $k$  or fewer members of  $\mathcal{F}$  have a common point. Then  $\mathcal{F}$  can be pierced by a multipoint.*


**Proof.** We use notation from the proof of Theorem 3.1. We apply Lemma 3.2 with  $t = 1$ , obtaining a set  $T$  with one point in each  $T_i$  such that all the  $2d$  vertices of the escape hypergraph  $\mathcal{H} = \mathcal{H}(T)$  have the same weight  $W$ . If  $W = 0$  we are done, so let us assume  $W > 0$ .

By the assumption on  $\mathcal{F}$ , every  $k$  edges of  $\mathcal{H}$  share a common vertex. We will prove the following claim for every  $\ell$ :

*if every  $\ell + 1$  edges of  $\mathcal{H}$  have at least  $m$  common vertices, then every  $\ell$  edges of  $\mathcal{H}$  have at least  $2m + 1$  common vertices.*

For  $\ell = k$ , the assumption holds with  $m = 1$ , and so by  $(k - 1)$ -fold application of this claim, we get that every edge of  $\mathcal{H}$  “intersects itself” in at least  $2^k - 1$  vertices, i.e.  $d > 2^k - 2$ . The claim thus implies the theorem.

The claim is proved by contradiction. Suppose that  $\mathcal{A} \subseteq \mathcal{H}$  is a set of  $\ell$  edges such that  $C = \bigcap \mathcal{A}$  has at most  $2m$  vertices. Recall that the vertices of  $\mathcal{H}$  are pairs  $(i, j)$ ,  $j \in [2]$ . Let  $\bar{C} = \{(i, 3 - j) : (i, j) \in C\}$  (note that  $C$  never contains both  $(i, 1)$  and  $(i, 2)$ , since no edge of  $\mathcal{H}$  does). By the assumption,  $\mathcal{A}$  plus any other edge together intersect in at least  $m$  vertices. Thus, any  $H \in \mathcal{H} \setminus \mathcal{A}$  contains at least  $m$  vertices of  $C$ , and consequently no more than  $m$  vertices of  $\bar{C}$ .

Let  $W$  be the total weight of the vertices in  $C$  and  $\bar{W}$  the total weight of the vertices in  $\bar{C}$ . The edges in  $\mathcal{A}$  contribute solely to  $W$ , while any other edge  $H$  contributes at least as much to  $W$  as to  $\bar{W}$ , and so  $W > \bar{W}$ . But this is impossible since all vertex weights are identical and  $|C| = |\bar{C}|$ . The claim, and Theorem 3.6 too, are proved. 

An interesting open problem is whether  $k = \lceil \log_2(d + 2) \rceil$  in Theorem 3.6 could be replaced by  $k = k_0$  for some constant  $k_0$  independent of  $d$ . The best known lower bound is  $k_0 \geq 3$ .

**Notes.** Tardos [Tar95] proved the optimal bound  $\tau \leq 2\nu$  for 2-intervals by a topological argument using the homology of suitable simplicial complexes. Kaiser’s argument [Kai97] is similar to the presented one, but he proves Lemma 3.2 using a rather advanced Borsuk–Ulam-type theorem of Ramos [Ram96] concerning continuous maps defined on products of spheres. The method with Brouwer’s theorem was used by Kaiser and Rabinovich [KR99] for a proof of Theorem 3.6.

Alon’s short proof [Alo98] of the bound  $\tau \leq 2d^2\nu$  for families of  $d$ -intervals applies a powerful technique developed in Alon and Kleitman [AK92]. For the so-called Hadwiger–Debrunner  $(p, q)$ -problem solved in the latter paper, the quantitative bounds are probably quite far from the truth. It would be interesting to find an alternative topological approach to that problem, which could perhaps lead to better bounds. See, for example, Hell [Hel].

The variant of the piercing problem for families of homogeneous  $d$ -intervals has been considered simultaneously with  $d$ -intervals ([GL85], [Tar95], [Kai97], [Alo98]). The upper bounds obtained for the homogeneous case are slightly worse:  $\tau \leq 3\nu$  for homogeneous 2-intervals,

which is tight, and  $\tau \leq (d^2 - d + 1)\nu$  for homogeneous  $d$ -intervals,  $d \geq 3$  [Kai97]. The reason for the worse bounds is that the escape hypergraph needs no longer be  $d$ -partite, and so Füredi's theorem [Für81] relating  $\nu$  to  $\nu^*$  gives a little worse bound (for  $d = 2$ , one uses a theorem of Lovász instead, asserting that  $\nu^* \leq \frac{3}{2}\nu$  for any graph).

Sgall's construction [Sga96] answered a problem raised by Wigderson in 1985. The title of Sgall's paper refers to a different, but essentially equivalent, formulation of the problem dealing with labeled tournaments.

Alon [Alo02] proved by the method of [Alo98] that if  $T$  is a tree and  $\mathcal{F}$  is a family subgraphs of  $T$  with at most  $d$  connected components, then  $\tau(\mathcal{F}) \leq 2d^2\nu(\mathcal{F})$ . More generally, he established a similar bound for the situation where  $T$  is a graph of bounded tree-width (on the other hand, if the tree-width of  $T$  is sufficiently large, then one can find a system of connected subgraphs of  $T$  with  $\nu = 1$  and  $\tau$  arbitrarily large, and so the tree-width condition is also necessary in this sense). A somewhat weaker bound for trees has been obtained independently by Kaiser [Kai98].

## Exercises

1. Let  $P$  and  $Q$  be convex polytopes. Show that there is a bijection between the nonempty faces of the Cartesian product  $P \times Q$  and all the products  $F \times G$ , where  $F$  is a nonempty face of  $P$  and  $G$  is a nonempty face of  $Q$ .
2. Show that the following “Brouwer-like” claim resembling Lemma 3.3 is *not* true: if  $f: B^n \rightarrow B^n$  is a continuous map of the  $n$ -ball such that the boundary of  $B^n$  is mapped surjectively onto itself, then  $f$  is surjective.
3. Prove the bound  $\tau(\mathcal{F}) \leq d^2\nu(\mathcal{F})$  for any family of *homogeneous*  $d$ -intervals (unions of  $d$  intervals on a single line). Hint: follow the proof for  $d$ -intervals above, but encode a candidate transversal  $T$  by a point of a simplex (rather than a product of simplices).

## 4 The fixed point theorems of Lefschetz, Smith, and Oliver

Fixed point theorems are “global-local tools”: from global information about a space (such as its homology) they derive local effects, such as the existence of special points where “something happens.”

Of course, in application to combinatorial problems we need to combine them with suitable “continuous-discrete tools”: from continuous effects, such as topological information about continuous maps of simplicial complexes, we have to find our way back to combinatorial information.

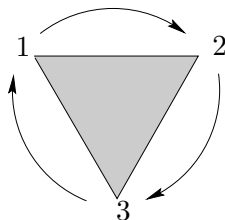
In addition to the usual game of graphs, posets, complexes and spaces, we will in the following exploit the deep topological effects<sup>4</sup> caused by symmetry, that is, by finite group actions.

A (finite) group  $G$  *acts* on a (finite) simplicial complex<sup>5</sup>  $K$  if each group element corresponds to a permutation of the vertices of  $K$ , where composition of group elements corresponds to composition of permutations, in such a way that  $g(A) := \{gv : v \in A\}$  is a face of  $K$  for all  $g \in G$  and for all  $A \in K$ . This action on the vertices is extended to the geometric realization of the complex  $K$ , so that  $G$  acts as a group of simplicial homeomorphisms  $g: \|K\| \rightarrow \|K\|$ .

The action is *faithful* if only the identity element in  $G$  acts as the identity permutation. In general, the set  $G_0 := \{g \in G : gv = v \text{ for all } v \in \text{vert}(K)\}$  is a normal subgroup of  $G$ . Hence we get that the quotient group  $G/G_0$  acts faithfully on  $K$ , and we usually only consider faithful actions. In this case, we can interpret  $G$  as a subgroup of the *symmetry group* of the complex  $K$ . The action is *vertex transitive* if for any two vertices  $v, w$  of  $K$  there is a group element  $g \in G$  with  $gv = w$ .

A *fixed point* (also known as *stable point*) of a group action is a point  $x \in \|K\|$  that satisfies  $gx = x$  for all  $g \in G$ . We denote the set of all fixed points by  $K^G$ . Note: this is not in general a subcomplex of  $K$ .

*Example 4.1.* Let  $K = 2^{[3]}$  be the complex of a triangle, and let  $G = \mathbb{Z}_3$  be the cyclic group (a proper subgroup of the symmetry group  $\mathcal{S}_3$ ), acting such that a generator cyclically permutes the vertices,  $1 \mapsto 2 \mapsto 3 \mapsto 1$ .



<sup>4</sup>In this section, we assume familiarity with more Algebra and Algebraic Topology than in other parts of these lecture notes, including some basic finite group theory, chain complexes, etc. However, this is a survey section, no detailed proofs will be given. Skim or skip, depending on your tastes and familiarity with these notions.

<sup>5</sup>See [Mat07] for a detailed discussion of simplicial complexes, their geometric realizations, etc. In particular, we use the notation  $\|K\|$  for the polyhedron (the geometric realization of a simplicial complex  $K$ ).



This is a faithful action; its fixed point set consists of the center of the triangle only—this is not a subcomplex of  $K$ , although it corresponds to a subcomplex of the barycentric subdivision  $\text{sd}(K)$ .

**Lemma 4.2** (Two barycentric subdivisions).

- (1) *After replacing  $K$  by its barycentric subdivision, we get that the fixed point set  $K^G$  is a subcomplex of  $K$ .*
- (2) *After replacing  $K$  by its barycentric subdivision once again, we even get that the quotient space  $\|K\|/G$  can be constructed from  $K$  by identifying all faces with their images under the action of  $G$ ; that is, the equivalence classes of faces of  $K$ , with the induced partial order, form a simplicial complex that is homeomorphic to the quotient space  $\|K\|/G$ .*

We leave the proof as an exercise. It is not difficult; for details and further discussion see Bredon [Bre72, Sect. III.1].

A powerful tool on our agenda is Hopf's trace theorem. Let  $V$  be any finite-dimensional vector space  $V$ , or a free abelian group of finite rank. When we consider an endomorphism  $g: V \rightarrow V$  then the *trace*  $\text{trace}(g)$  is the sum of the diagonal elements of the matrix that represents  $g$ . The trace is independent of the basis chosen for  $V$ . In the case when  $V$  is a free abelian group, then  $\text{trace}(g)$  is an integer.

**Theorem 4.3** (The Hopf trace theorem). *Let  $f: \|K\| \rightarrow \|K\|$  be a self-map, and denote by  $f_{\#i}$  resp.  $f_{*i}$  the maps that  $f$  induces on  $i$ -dimensional chain groups resp. homology groups.*

*Using an arbitrary field of coefficients  $k$ , one has*

$$\sum_i (-1)^i \text{trace}(f_{\#i}) = \sum_i (-1)^i \text{trace}(f_{*i}).$$

*The same identity holds if we use integer coefficients, and compute the traces for homology in the quotients  $H_i(K, \mathbb{Z})/T_i(K, \mathbb{Z})$  of the homology groups modulo their torsion subgroups; these quotients are free abelian groups.*

The proof for this uses the definition of simplicial homology, and simple linear algebra; we refer to Munkres [Mun84, Thm. 22.1] or Bredon [Bre93, Sect. IV.23].

For an arbitrary coefficient field  $k$ , we define the *Lefschetz number* of the map  $f: \|K\| \rightarrow \|K\|$  as

$$L_k(f) := \sum_i (-1)^i \text{trace}(f_{*i}) \in k.$$

Similarly, taking integral homology modulo torsion, we define the *integral Lefschetz number* as

$$L(f) := \sum_i (-1)^i \text{trace}(f_{*i}) \in \mathbb{Z}.$$

The universal coefficient theorems imply that one always has  $L_{\mathbb{Q}}(f) = L(f)$ : thus the integral Lefschetz number  $L(f)$  can be computed in rational homology, but it is an integer.

The *Euler characteristic* of a complex  $K$  coincides with the Lefschetz number of the identity map  $\text{id}_K: \|K\| \rightarrow \|K\|$ ,

$$\chi(K) = L(\text{id}_K), \text{ where } \text{trace}((\text{id}_K)_{*i}) = \beta_i(K).$$

Thus the Hopf trace theorem yields that the Euler-characteristic of a finite simplicial complex  $K$  can be defined resp. computed without a reference to homology, simply as the alternating sum of the face numbers of the complex  $K$ , where  $f_i = F_i(K)$  denotes the number of  $i$ -dimensional faces of  $K$ :

$$\chi(K) := f_0(K) - f_1(K) + f_2(K) - \cdots.$$


This is then a finite sum that ends with  $(-1)^d f_d(K)$  if  $K$  has dimension  $d$ . Thus the Hopf trace theorem applied to the identity map just reproduces the Euler–Poincaré formula. This proves, for example, the  $d$ -dimensional Euler polyhedron formula, not only for polytopes, but also for general spheres, shellable or not (see Ziegler [Zie98]).

For us the main consequence of the trace formula is the following theorem.

**Theorem 4.4** (The Lefschetz fixed point theorem). *Let  $K$  be a finite simplicial complex, and  $k$  an arbitrary field. If a self-map  $f: \|K\| \rightarrow \|K\|$  has Lefschetz number  $L_k(f) \neq 0$ , then  $f$  and any map homotopic to  $f$  have a fixed point.*

*In particular, if  $K$  is  $\mathbb{Z}_p$ -acyclic for some prime  $p$ , then every continuous map  $f: \|K\| \rightarrow \|K\|$  has a fixed point.*

**Proof** (Sketch). For a finite simplicial complex  $K$ , the polyhedron  $\|K\|$  is compact. So if  $f$  does not have a fixed point, then one has  $\varepsilon > 0$  such that  $|f(x) - x| > \varepsilon$  for all  $x \in K$ . Now take a subdivision into simplices of diameter smaller than  $\varepsilon$ , and a simplicial approximation of error smaller than  $\varepsilon/2$ , so that the simplicial approximation does not have a fixed point, either.

Now apply the trace theorem, where the induced map  $f_{*0}$  in 0-dimensional homology is the identity. 

Note that Brouwer’s Fixed Point Theorem 1.4 is the special case of Theorem 4.4 when  $K$  triangulates a ball.


For a reasonably large class of spaces, a converse to the Lefschetz Fixed Point Theorem is also true: If  $L(f) = 0$ , then  $f$  is homotopic to a map without fixed points. See Brown [Bro71, Chap. VIII].

“Smith Theory” was started by P. A. Smith [Smi41] in the thirties—we refer to Bredon [Bre72, Chapter III] for a nice textbook treatment. Smith Theory analyses finite group actions on compact spaces (such as finite simplicial complexes), providing relations between the structure of the group to its possible fixed point sets. Here is one key result.

**Theorem 4.5** (Smith [Smi38]). *If  $P$  is a  $p$ -group (that is, a finite group of order  $|P| = p^t$  for a prime  $p$  and some  $t > 0$ ), acting on a complex  $K$  that is  $\mathbb{Z}_p$ -acyclic, then the fixed point set  $K^P$  is  $\mathbb{Z}_p$ -acyclic as well. In particular, it is not empty.*

**Proof** (Sketch). The key is that, with the preparations of Lemma 4.2, the maps that  $f$  induces on the chain groups (with  $\mathbb{Z}_p$  coefficients) nicely restrict to the chain groups on the fixed point set  $K^P$ . Passing to traces and using the Hopf trace theorem, one can derive that  $K^P$  is non-empty.

A more detailed analysis leads to the “transfer isomorphism” in homology, which proves that  $K^P$  must be acyclic.

See Bredon [Bre72, Thm. III.5.2] and Oliver [Oli75, p. 157], and also de Longueville [dL13, Appendix D and E]. 

On the combinatorial side, one has an Euler characteristic relation due to Floyd [Flo52] [Bre72, Sect. III.4]:

$$\chi(K) + (p-1)\chi(K^{\mathbb{Z}_p}) = p\chi(K/\mathbb{Z}_p).$$

It implies that if  $P$  is a  $p$ -group (in particular, if  $P = \mathbb{Z}_p$ ), then


$$\chi(K^P) \equiv \chi(K) \pmod{p},$$

using induction on  $t$ , where  $|P| = p^t$ .

**Theorem 4.6** (Oliver [Oli75, Lemma I]). *If  $G = \mathbb{Z}_n$  is a cyclic group, acting on a  $\mathbb{Q}$ -acyclic complex  $K$ , then the action has a fixed point.*

*In fact, in this case the fixed point set  $K^G$  has the Euler characteristic of a point,  $\chi(K^G) = 1$ .*

**Proof.** The first statement follows directly from the Lefschetz fixed point theorem: any cyclic group is generated by a single element  $g$ , this element has a fixed point, this fixed point of  $g$  is also a fixed point of all powers of  $g$ , and hence of the whole group  $G$ .

For the second part, take  $p^t$  to be a maximal prime power that divides  $n$ , consider the corresponding subgroup isomorphic to  $\mathbb{Z}_{p^t}$ , and use induction on  $t$  and the transfer homomorphism, as for the previous proof. 

Unfortunately, results like these may give an overly optimistic impression of the generality of fixed point theorems for acyclic complexes. There are fixed point free finite group actions on balls: examples were constructed by Floyd & Richardson and others; see Bredon [Bre72, Sect. I.8].

On the positive side we have the following result due to Oliver, which will play a central role in the following section.

**Theorem 4.7** (Oliver’s Theorem I [Oli75, Prop. I]). *If  $G$  has a normal subgroup  $P \triangleleft G$  that is a  $p$ -group, such that the quotient  $G/P$  is cyclic, acting on a complex  $K$  that is  $\mathbb{Z}_p$ -acyclic, then the fixed point set  $K^G$  is  $\mathbb{Z}_p$ -acyclic as well. In particular, it is not empty.*

This is as much as we will need in this chapter. Oliver proved, in fact, a more general and complete theorem that includes a converse.

**Theorem 4.8** (Oliver’s Theorem II [Oli75]). *Let  $G$  be a finite group. Every action of  $G$  on a  $\mathbb{Z}_p$ -acyclic complex  $K$  has a fixed point if and only if  $G$  has the following structure:*

*$G$  has normal subgroups  $P \triangleleft Q \triangleleft G$  such that  $P$  is a  $p$ -group,  $G/Q$  is a  $q$ -group (for a prime  $q$  that need not be distinct from  $p$ ), and the quotient  $Q/P$  is cyclic.*

*In this situation one always has  $\chi(K^G) \equiv 1 \pmod{q}$ .*

## Notes

The Lefschetz–Hopf fixed point theorem was announced by Lefschetz for a restricted class of complexes in 1923, with details appearing three years later. The first proof for the general version was by Hopf in 1929. There are generalizations, for example to Absolute Neighborhood Retracts; see Bredon [Bre93, Cor. IV.23.5] and Brown [Bro71, Chap. III]. We refer to Brown’s book [Bro71].

## 5 Evasiveness

### 5.1 A general model

Evasiveness appears in different versions for graphs, digraphs and bipartite graphs. Therefore, we start with a general model that contains and, perhaps, explains them all.

**Definition 5.1** (Argument complexity of a set system; evasiveness). In the following, we are concerned with a fixed, known set system  $\mathcal{F} \subseteq 2^E$ , and with the complexity of deciding whether some set  $A \subseteq E$  is in the set system. Here our “model of computation” is such that

**given, and known**, is a set system  $\mathcal{F} \subseteq 2^E$ , where  $E$  is fixed,  $|E| = m$ .

On the other hand, there is a

**fixed, but unknown** subset  $A \subseteq E$ . We have to

**decide** whether  $A \in \mathcal{F}$ , using only

**questions** of the type “Is  $e \in A$ ?”

(It is assumed that we always get correct answers YES or NO. We only count the *number* of questions that are needed in order to reach the correct conclusion: it is assumed that it is not difficult to decide whether  $e \in A$ . You can assume that some “oracle” that knows both  $A$  and  $\mathcal{F}$  is answering.)

The *argument complexity*  $c(\mathcal{F})$  of the set system  $\mathcal{F}$  is the number of elements of the ground set  $E$  that we have to test in the worst case—with the optimal strategy.

Clearly  $0 \leq c(\mathcal{F}) \leq m$ . The set system  $\mathcal{F}$  is *trivial* if  $c(\mathcal{F}) = 0$ : then no questions need to be asked; this can only be the case if  $\mathcal{F} = \{\emptyset\}$  or if  $\mathcal{F} = 2^E$ . (Otherwise  $\mathcal{F}$  is *non-trivial*.)

The set system  $\mathcal{F}$  is *evasive* if  $c(\mathcal{F}) = m$ , that is, if even with an optimal strategy one has to test all the elements of  $E$  in the worst case.

For example, if  $\mathcal{F} = \{\emptyset\}$ , then  $c(\mathcal{F}) = m$ : if we again and again get the answer NO, then we have to test all the elements to be sure that  $A = \emptyset$ . So  $\mathcal{F} = \{\emptyset\}$  is an evasive set system: “being empty” is an evasive set property.

### 5.2 Complexity of graph properties

**Definition 5.2** (Graph properties). For this we consider graphs on a fixed vertex set  $V = [n]$ . Loops and multiple edges are excluded. Thus any graph  $G = (V, A)$  is determined by its edge set  $A$ , which is a subset of the set  $E = \binom{[n]}{2}$  of “potential edges.”

We identify a *property*  $\mathcal{P}$  of graphs with the family of graphs that have the property  $\mathcal{P}$ , and thus with the set family  $\mathcal{F}(\mathcal{P}) \subseteq 2^E$  given by

$$\mathcal{F}(\mathcal{P}) := \{A \subseteq E : ([n], A) \text{ has property } \mathcal{P}\}.$$

Furthermore, we will consider only graph properties that are isomorphism invariant; that is, properties of abstract graphs that are preserved under renumbering the vertices.

A graph property is *evasive* if the associated set system is evasive, and otherwise it is *non-evasive*.

With the symmetry condition of Definition 5.2, we would accept “being connected”, “being planar,” “having no isolated vertices,” and “having even vertex degrees” as graph properties. However, “vertex 1 is not isolated,” “123 is a triangle,” and “there are no edges between odd-numbered vertices” are not graph properties.

*Examples 5.3* (Graph properties). For the following properties of graphs on  $n$  vertices we can easily determine the argument complexity.

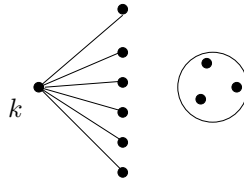
**Having no edge:** Clearly we have to check every single  $e \in E$  in order to be sure that it is not contained in  $A$ , so this property is evasive: its argument complexity is  $c(\mathcal{F}) = m = \binom{n}{2}$ .

**Having at most  $k$  edges:** Let us assume that we ask questions, and the answer we get is YES for the first  $k$  questions, and then we get NO answers for all further questions, except for possibly the last one. Assuming that  $k < m$ , this implies that the property is evasive. Otherwise, for  $k \geq m$ , the property is trivial.

**Being connected:** This property is evasive for  $n \geq 2$ . Convince yourself that for any strategy, a sequence of “bad” answers can force you to ask all the questions.

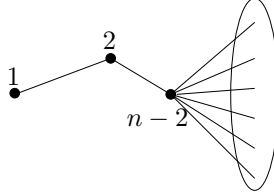
**Being planar:** This property is trivial for  $n \leq 4$  but evasive for  $n \geq 5$ . In fact, for  $n = 5$  one has to ask all the questions (in arbitrary order), and the answer will be  $A \in \mathcal{F}$  unless we get a YES answer for all the questions—including the last one. This is, however, not at all obvious for  $n > 5$ : it was claimed by Hopcroft & Tarjan [HT74], and proved by Best, Van Emde Boas & Lenstra [BvEBL74, Example 2] [Bol78, p. 408].

**A large star:** Let  $\mathcal{P}$  be the property of being a disjoint union of a star  $K_{1,n-4}$  and an arbitrary graph on 3 vertices, and let  $\mathcal{F}$  be the corresponding set system.



Then  $c(\mathcal{F}) < \binom{n}{2}$  for  $n \geq 7$ . For  $n \geq 12$  we can easily see this, as follows. Test all the  $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$  edges  $\{i, j\}$  with  $i \leq \lfloor \frac{n}{2} \rfloor < j$ . That way we will find exactly one vertex  $k$  with at least  $\lfloor \frac{n}{2} \rfloor - 3 \geq 3$  neighbors (otherwise property  $\mathcal{P}$  cannot be satisfied): that vertex  $k$  has to be the center of the star. We test all other edges adjacent to  $k$ : we must find that  $k$  has exactly  $n - 4$  neighbors. Thus we have identified three vertices that are not neighbors of  $k$ : at least one of the edges between those three has not been tested. We test all other edges to check that  $([n], A)$  has property  $\mathcal{P}$ . (This property was found by L. Carter [BvEBL74, Example 16].)

**Being a scorpion graph:** A *scorpion graph* is an  $n$ -vertex graph that has one vertex of degree 1 adjacent to a vertex of degree 2 whose other neighbor has degree  $n - 2$ . We leave it as an (instructive!) exercise to check that “being a scorpion graph” is not evasive if  $n$  is large: in fact, Best, van Emde Boas & Lenstra [BvEBL74, Example 18] [Bol78, p. 410] have shown that  $c(\mathcal{F}) \leq 6n$ .



From these examples it may seem that most “interesting” graph properties are evasive. In fact, many more examples of evasive graph properties can be found in Bollobás [Bol78, Sect. VIII.1], alongside with techniques to establish that graph properties are evasive, such as Milner & Welsh’s “simple strategy” [Bol78, p. 406].

Why is this model of interest? Finite graphs (similarly for digraphs and bipartite graphs) can be represented in different types of *data structures* that are not at all equivalent for algorithmic applications. For example, if a finite graph is given by an adjacency list, then one can decide fast (“in linear time”) whether the graph is planar, e.g. using an old algorithm of Hopcroft & Tarjan [HT74]; see also Mehlhorn [Meh84, Sect. IV.10] and [MM96]. Note that such a planar graph has at most  $3n - 6$  edges (for  $n \geq 3$ ).

However, assume that a graph is given in terms of its adjacency matrix

$$M(G) = \left( m_{ij} \right)_{1 \leq i, j \leq n} \in \{0, 1\}^{n \times n},$$

where  $m_{ij} = 1$  means that  $\{i, j\}$  is an edge of  $G$ , and  $m_{ij} = 0$  says that  $\{i, j\}$  is not an edge. Here  $G$  is faithfully represented by the set of all  $\binom{n}{2}$  superdiagonal entries (with  $i < j$ ). Then one possibly has to inspect a large part of the matrix until one has enough information to decide whether the graph in question is planar. In fact, if  $\mathcal{F} \subseteq 2^E$  is the set system corresponding to all planar graphs, then  $c(\mathcal{F})$  is exactly the number of superdiagonal matrix entries that every algorithm for planarity testing has to inspect in the worst case.

The statement that “being planar” is evasive (for  $n \geq 5$ ) thus translates into the fact that every planarity testing algorithm that starts from an adjacency matrix needs to read at least  $\binom{n}{2}$  bits of the input, and hence its running time is bounded from below by  $\binom{n}{2} = \Omega(n^2)$ . This means that such an algorithm—such as the one considered by Fisher [Fis66]—cannot run in linear time, and thus cannot be efficient.

**Definition 5.4** (Digraph properties; bipartite graph properties).

- (1) For digraph properties we again use the fixed vertex set  $V = [n]$ . Loops and parallel edges are excluded, but anti-parallel edges are allowed. Thus any digraph  $G = (V, A)$  is determined by its arc set  $A$ , which is a subset of the

set  $E'$  of all  $m := n^2 - n$  “potential arcs” (corresponding to the off-diagonal entries of an  $n \times n$  adjacency matrix).

A *digraph property* is a property of digraphs  $([n], A)$  that is invariant under relabelling of the vertex set. Equivalently, a digraph property is a family of arc sets  $\mathcal{F} \subseteq 2^{E'}$  that is symmetric under the action of  $\mathcal{S}_n$  that acts by renumbering the vertices (and renumbering all arcs correspondingly). A digraph property is *evasive* if the associated set system is evasive, otherwise it is *non-evasive*.

- (2) For bipartite graph properties we use a fixed vertex set  $V \uplus W$  of size  $m + n$ , and use  $E'' := V \times W$  as the set of potential edges. A *bipartite graph property* is a property of graphs  $(V \cup W, A)$  with  $A \subseteq E''$  that is preserved under renumbering the vertices in  $V$ , and also under permuting the vertices in  $W$ . Equivalently, a bipartite graph property on  $V \times W$  is a set system  $\mathcal{F} \subseteq 2^{V \times W}$  that is stable under the action of the automorphism group  $\mathcal{S}_n \times \mathcal{S}_m$  that acts transitively on  $V \times W$ .

*Examples 5.5* (Digraph properties). For the following digraph properties on  $n$  vertices we can determine the argument complexity.

**Having at most  $k$  arcs:** Again, this is clearly evasive with  $c(\mathcal{F}) = m$  if  $k < m = n^2 - n$ , and trivial otherwise.

**Having a sink:** A *sink* in a digraph on  $n$  vertices is a vertex  $k$  for which all arcs going into  $k$  are present, but no arc leaves  $k$ , that is, a vertex of out-degree  $\delta^+(v) = 0$ , and in-degree  $\delta^-(v) = n - 1$ . Let  $\mathcal{F}$  be the set system of all digraphs on  $n$  vertices that have a sink. It is easy to see that  $c(\mathcal{F}) \leq 3n - 4$ . In particular, for  $n \geq 3$  “having a sink” is a non-trivial but non-evasive digraph property.

In fact, if we test whether  $(i, j) \in A$ , then either we get the answer YES, then  $i$  is not a sink, or we get the answer NO, then  $j$  is not a sink. So, by testing arcs between pairs of vertices that “could be sinks,” after  $n - 1$  questions we are down to one single “candidate sink”  $k$ . At this point at least one arc adjacent to  $k$  has been tested. So we need at most  $2n - 3$  further questions to test whether it is a sink.

*Remark 5.6* (History: The Aanderaa–Rosenberg Conjecture). Originally, Arnold L. Rosenberg had conjectured that all non-trivial digraph properties have quadratic argument complexity, that is, that there is a constant  $\gamma > 0$  such that for all non-trivial properties of digraphs on  $n$  vertices one has  $c(\mathcal{F}) \geq \gamma n^2$ . However, S. Aanderaa found the counter-example (for digraphs) of “having a sink” [BvEBL74, Example 15] [RV78, p. 372]. We have also seen that “being a scorpion graph” is a counter-example for graphs.

Hence Rosenberg modified the conjecture: at least all *monotone* graph properties, that is, properties that are preserved under deletion of edges, should have quadratic argument complexity. This is the statement of the *Aanderaa–Rosenberg conjecture* [Ros73]. Richard Karp considerably sharpened the statement, as follows.



**Conjecture 5.7** (Evasiveness conjecture, Karp [Ros73]). *Every non-trivial monotone graph property or digraph property is evasive.*

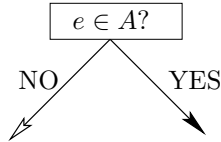
We will prove this below for graphs and digraphs in the special case when  $n$  is a prime power; from this one can derive the Aanderaa–Rosenberg conjecture, with  $\gamma \approx \frac{1}{4}$ . Similarly, we will prove that monotone properties of bipartite graphs on a fixed ground set  $V \cup W$  are evasive (without any restriction on  $|V| = m$  and  $|W| = n$ ). However, we first return to the more general setting of set systems.

### 5.3 Decision trees

Any strategy to determine whether an (unknown) set  $A$  is contained in a (known) set system  $\mathcal{F}$ —as in Definition 5.1—can be represented in terms of a decision tree of the following form.

**Definition 5.8.** A *decision tree* is a rooted, planar, binary tree whose leaves are labelled “YES” or “NO,” and whose internal nodes are labelled by questions (here they are of the type “ $e \in A$ ?”). Its edges are labelled by answers: we will represent them so that the edges labelled “YES” point to the right child, and the “NO” edges pointing to the left child.

A *decision tree for  $\mathcal{F} \subseteq 2^E$*  is a decision tree such that starting at the root with an arbitrary  $A \subseteq E$ , and going to the right resp. left child depending on whether the question at an internal node we reach has answer YES or NO, we always reach a leaf that correctly answers the question “ $A \in \mathcal{F}$ ?”.



The root of a decision tree is at *level* 0, and the children of a node at level  $i$  have level  $i + 1$ . The *depth* of a tree is the greatest  $k$  such that the tree has a vertex at level  $k$  (a leaf).

We assume (without loss of generality) that the trees we consider correspond to strategies where we never ask the same question twice.

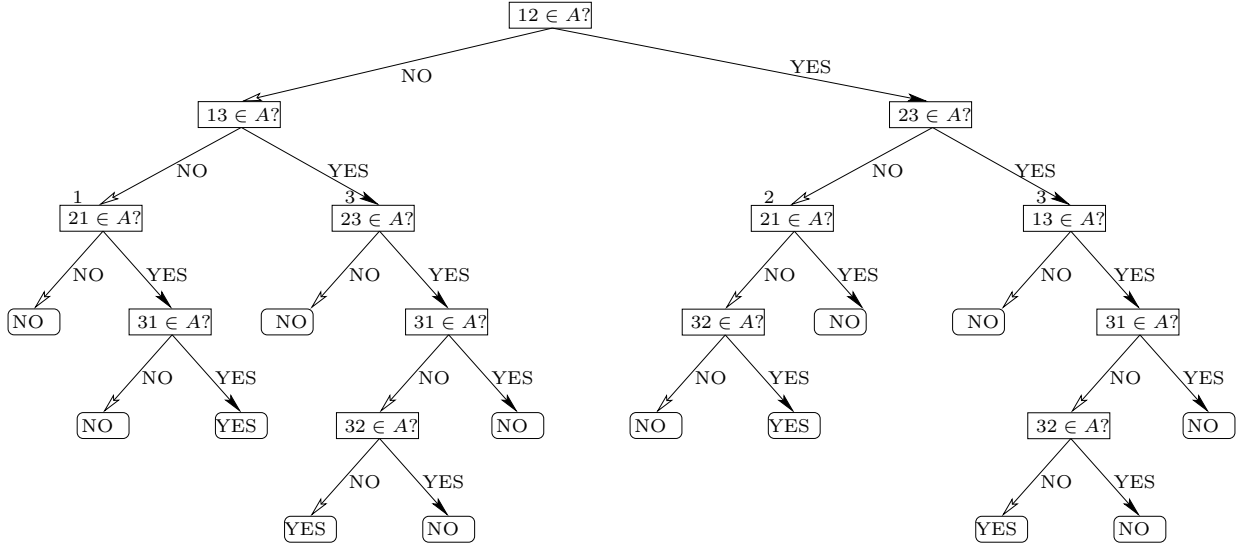
A decision tree for  $\mathcal{F}$  is *optimal* if it has the smallest depth among all decision trees for  $\mathcal{F}$ ; that is, if it leads us to ask the smallest number of questions for the worst possible input.

Let us consider an explicit example.

The following figure represents an optimal algorithm for the “sink” problem on digraphs with  $n = 3$  vertices. This has a ground set  $E = \{12, 21, 13, 31, 23, 32\}$  of size  $m = 6$ .

The algorithm first asks, in the root node at level 0, whether  $12 \in A?$ . In case the answer is YES (so we know that 1 is not a sink), it branches to the right, leading to a question node at level 1 that asks whether  $23 \in A?$ , etc. In case the answer to the question  $12 \in A?$  is NO (so we know that 2 is not a sink), it branches to the left, leading to a question node at level 1 that asks whether  $13 \in A?$ , etc.

For every possible input  $A$  (there are  $2^6 = 32$  different ones), after two questions we have identified a unique “candidate sink”; after not more than 5 question nodes one arrives at a leaf node that correctly answers the question whether the graph  $(V, A)$  has a sink node: YES or NO. (The number of the unique candidate is noted next to each node at level 2.)



For each node (leaf or inner) of level  $k$ , there are exactly  $2^{m-k}$  different inputs that lead to this node. This proves the following lemma.

**Lemma 5.9.** *The following are equivalent:*

- $\mathcal{F}$  is non-evasive.
- The optimal decision trees  $T_{\mathcal{F}}$  for  $\mathcal{F}$  have depth smaller than  $m$ .
- Every leaf of an optimal decision tree  $T_{\mathcal{F}}$  is reached by at least two distinct inputs.

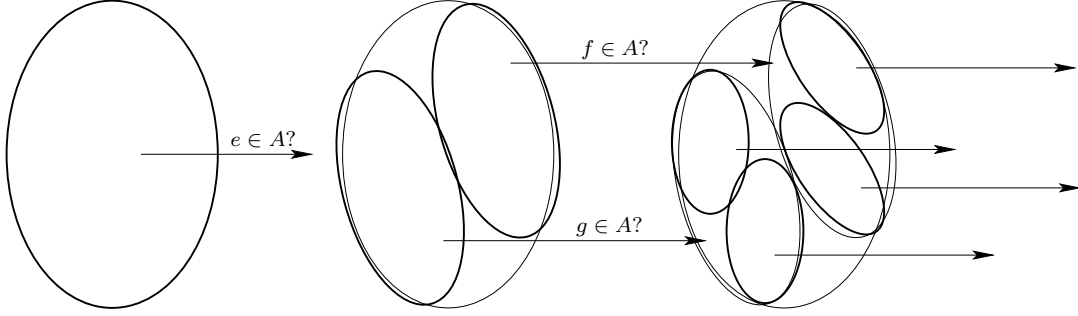
**Corollary 5.10.** *If  $\mathcal{F}$  is non-evasive, then  $|\mathcal{F}|$  is even.*

This can be used to show, for example, that the directed graph property “has a directed cycle” is evasive [BvEBL74, Example 4].

Another way to view a (binary) decision tree algorithm is as follows. In the beginning, we do not know anything about the set  $A$ , so we can view the collection of possible sets as the complete boolean algebra of all  $2^m$  subsets of  $E$ .

In the first node (at “level 0”) we ask a question of the type “ $e \in A?$ ”; this induces a subdivision of the boolean algebra into two halves, depending on whether we get answer YES or NO. Each of the halves is an interval of length  $m - 1$  of the boolean algebra  $(2^E, \subseteq)$ .

At level 1 we ask a new question, depending on the outcome of the first question. Thus we *independently* bisect the two halves of level 0, getting four pieces of the boolean algebra, all of the same size.



This process is iterated. It stops—we do not need to ask a further question—on the parts which we create that either contain only sets that are in  $\mathcal{F}$  (this yields a YES-leaf) or that contain only sets not in  $\mathcal{F}$  (corresponding to NO-leaves).

Thus the final result is a special type of partition of the boolean algebra into intervals. Some of them are YES intervals, containing only sets of  $\mathcal{F}$ , all the others are NO-intervals, containing no sets from  $\mathcal{F}$ . If the property in question is monotone, then the union of the YES intervals (i. e., the set system  $\mathcal{F}$ ) forms an *ideal* in the boolean algebra, that is, a “down-closed” set such that with any set that it contains it must also contain all its subsets.

Let  $p_{\mathcal{F}}(t)$  be the generating function for the set system  $\mathcal{F}$ , that is, the polynomial

$$p_{\mathcal{F}}(t) := \sum_{A \in \mathcal{F}} t^{|A|} = f_{-1} + t f_0 + t^2 f_1 + t^3 f_2 + \dots$$

where  $f_i = |\{A \in \mathcal{F} : |A| = i + 1\}|$ .

**Proposition 5.11.**

$$(1 + t)^{m - c(\mathcal{F})} \mid p_{\mathcal{F}}(t).$$

**Proof.** Consider one interval  $\mathcal{I}$  in the partition of  $2^E$  that is induced by any optimal algorithm for  $\mathcal{F}$ . If the leaf, at level  $k$ , corresponding to the interval is reached through a sequence of  $k_Y$  YES-answers and  $k_N$  NO-answers (with  $k_Y + k_N = k$ ), then this means that there are sets  $A_Y \subseteq E$  with  $|A_Y| = k_Y$  and  $A_N \subseteq E$  with  $|A_N| = k_N$ , such that

$$\mathcal{I} = \{A \subseteq E : A_Y \subseteq A \subseteq E \setminus A_N\}.$$


In other words, the interval  $\mathcal{I}$  contains all sets that give YES-answers when asked about any of the  $k_Y$  elements of  $A_Y$ , NO-answers when asked about any of the  $k_N$  elements of  $A_N$ , while the  $m - k_Y - k_N$  elements of  $E \setminus (A_Y \cup A_N)$

may or may not be contained in  $A$ . Thus the interval  $\mathcal{I}$  has size  $2^{m-k_Y-k_N}$ , and its counting polynomial is

$$p_{\mathcal{I}}(t) := \sum_{A \in \mathcal{I}} t^{|A|} = t^{k_Y} (1+t)^{m-k_Y-k_N}.$$


Now the complete set system  $\mathcal{F}$  is a disjoint union of the intervals  $\mathcal{I}$ , and we get

$$p_{\mathcal{F}}(t) = \sum_{\mathcal{I}} p_{\mathcal{I}}(t).$$

In particular, for an optimal decision tree we have  $k_Y + k_N = k \leq c(\mathcal{F})$  and thus  $m - c(\mathcal{F}) \leq m - k_Y - k_N$  at every leaf of level  $k$ , which means that all the summands  $p_{\mathcal{I}}(t)$  have a common factor of  $(1+t)^{m-c(\mathcal{F})}$ . 

**Corollary 5.12.** *If  $\mathcal{F}$  is non-evasive, then  $|\mathcal{F}^{even}| = |\mathcal{F}^{odd}|$ , that is,*

$$-f_{-1} + f_0 - f_1 + f_2 \mp \dots = 0.$$

**Proof.** Use Lemma 5.9, and put  $t = -1$ . 

We can now draw the conclusion, based only on simple counting, that most set families are evasive. This cannot of course be used to settle any specific cases, but it can at least make the various evasiveness conjectures seem more plausible.


**Corollary 5.13.** *Asymptotically, almost all set families  $\mathcal{F}$  are evasive.*

**Proof.** The number of set families  $\mathcal{F} \subseteq 2^E$  such that

$$\#\{A \in \mathcal{F} \mid \#A \text{ odd}\} = \#\{A \in \mathcal{F} \mid \#A \text{ even}\} = k$$

is  $\binom{2^{m-1}}{k}^2$ . Hence, using Stirling's estimate of factorials,

$$\text{Prob}(\mathcal{F} \text{ non-evasive}) \leq \frac{\sum_{k=0}^{2^{m-1}} \binom{2^{m-1}}{k}^2}{2^{2^m}} = \frac{\binom{2^m}{2^{m-1}}}{2^{2^m}} \sim \frac{1}{\sqrt{\pi 2^{m-1}}} \rightarrow 0,$$

as  $m \rightarrow \infty$ . 


**Conjecture 5.14** (The “Generalized Aanderaa–Rosenberg Conjecture”, Rivest & Vuillemin [RV76]). *If  $\mathcal{F} \subseteq 2^E$ , with symmetry group  $G \subseteq \mathcal{S}_E$  that is transitive on the ground set  $E$ , and if  $\emptyset \in \mathcal{F}$  but  $E \notin \mathcal{F}$ , then  $\mathcal{F}$  is evasive.*

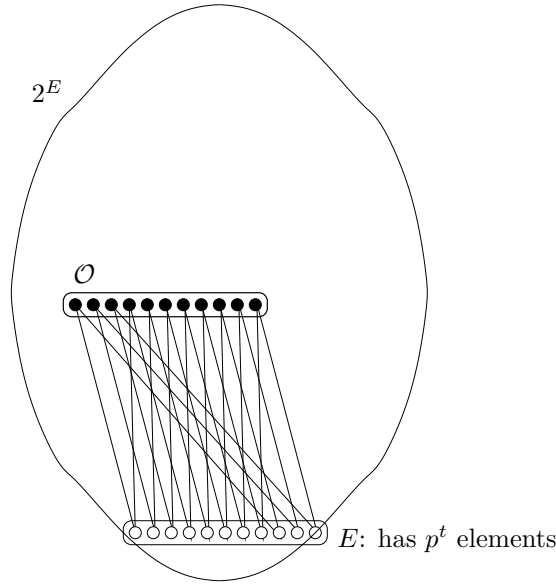
Note that for this it is *not* assumed that  $\mathcal{F}$  is monotone. However, the assumption that  $\emptyset \in \mathcal{F}$  but  $E \notin \mathcal{F}$  is satisfied neither by “being a scorpion” nor by “having a sink.”

**Proposition 5.15** (Rivest & Vuillemin [RV76]). *The Generalized Aanderaa–Rosenberg Conjecture 5.14 holds if the size of the grounds set is a prime power,  $|E| = p^t$ .*

**Proof.** Let  $\mathcal{O}$  be any  $k$ -orbit of  $G$ , that is, a collection of  $k$ -sets  $\mathcal{O} \subseteq \mathcal{F}$  on which  $G$  acts transitively. While every set in  $\mathcal{O}$  contains  $k$  elements  $e \in E$ , we know from transitivity that every element of  $E$  is contained in the same number, say  $d$ , of sets of the orbit  $\mathcal{O}$ . Thus, double-counting the edges of the bipartite graph on the vertex set  $E \uplus \mathcal{O}$  defined by “ $e \in A$ ” (displayed in the figure below) we find that  $k|\mathcal{O}| = d|E| = dp^t$ . Thus for  $0 < k < p^t$  we have that  $p$  divides  $|\mathcal{O}|$ , while  $\{\emptyset\}$  is one single “trivial” orbit of size 1, and  $k = p^t$  doesn’t appear. Hence we have

$$-f_{-1} + f_0 - f_1 + f_2 \mp \dots \equiv -1 \pmod{p},$$

which implies evasiveness by Corollary 5.12. 



**Proposition 5.16** (Illies [Ill78]). *The Generalized Aanderaa–Rosenberg Conjecture 5.14 fails for  $n = 12$ .*


**Proof.** Here is Illies’ counterexample: take  $E = \{1, 2, 3, \dots, 12\}$ , and let the cyclic group  $G = \mathbb{Z}_{12}$  permute the elements of  $E$  with the obvious cyclic action.

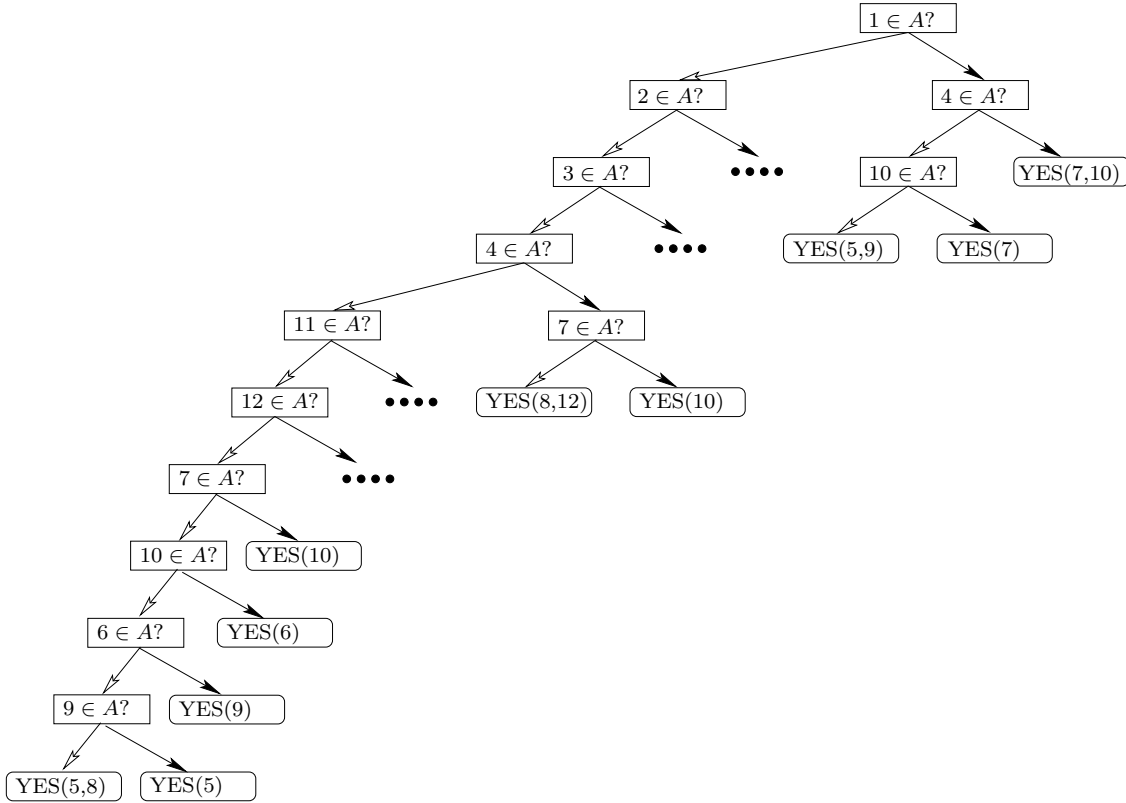
Take  $\mathcal{F}_I \subseteq 2^E$  to be the following system of sets

- $\emptyset$ , so we have  $f_{-1} = 1$
- $\{1\}$  and all images under  $\mathbb{Z}_{12}$ , that is, all singleton sets:  $f_0 = 12$ ,
- $\{1, 4\}$  and  $\{1, 5\}$  and all images under  $\mathbb{Z}_{12}$ , so  $f_1 = 12 + 12 = 24$ ,
- $\{1, 4, 7\}$  and  $\{1, 5, 9\}$  and all their  $\mathbb{Z}_{12}$ -images, so  $f_2 = 12 + 4 = 16$ ,
- $\{1, 4, 7, 10\}$  and their  $\mathbb{Z}_{12}$ -images, so  $f_3 = 3$ .

An explicit decision tree of depth 11 for this  $\mathcal{F}_I$  is given in our figure below. Here the *pseudo-leaf* “YES(7,10)” denotes a decision tree where we check all elements  $e \in E$  that have not been checked before, other than the elements 7 and 10. If none of them is contained in  $A$ , then the answer is YES (irrespective

of whether  $7 \in A$  or  $10 \in A$ ), otherwise the answer is NO. The fact that two elements need not be checked means that this branch of the decision tree denoted by this “pseudo-leaf” does not go beyond depth 10. Similarly, a pseudo-leaf of the type “YES(7)” represents a subtree of depth 11.

Thus the following figure completes the proof. Here dots denote subtrees that are analogous to the ones just above. 



Note that Illies' example is not monotone: for example, we have  $\{1, 4, 7\} \in \mathcal{F}_I$ , but  $\{1, 7\} \notin \mathcal{F}_I$ .

## 5.4 Monotone systems

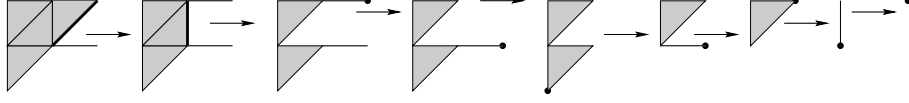
We now concentrate on the case where  $\mathcal{F}$  is closed under taking subsets, that is,  $\mathcal{F}$  is an abstract simplicial complex, which we also denote by  $K := \mathcal{F}$ . In this setting, the symmetry group acts on  $K$  as a group of simplicial homeomorphisms. If  $\mathcal{F}$  is a graph or digraph property, then this means that the action of  $G$  is transitive on the vertex set  $E$  of  $K$ , which corresponds to the edge set of the graph in question. Again we denote the cardinality of the ground set (the vertex set of  $K$ ) by  $|E| = m$ .

A complex  $K \subseteq 2^E$  is *collapsible* if it can be reduced to a one-point complex (equivalently, to a simplex) by steps of the form

$$K \longrightarrow K \setminus \{A \in K : A_0 \subseteq A \subseteq A_1\}$$

$\emptyset \subset A_0 \subset A_1$  are faces of  $K$  with  $\emptyset \neq A_0 \neq A_1$ , where  $A_1$  is the *unique* maximal element of  $K$  that contains  $A_0$ .

Our figure illustrates a sequence of collapses that reduces a 2-dimensional complex to a point. In each case the face  $A_0$  that is contained in a unique maximal face is drawn fattened.





**Theorem 5.17.** *We have the following implications:*

$K \text{ is a cone} \implies K \text{ is non-evasive} \implies K \text{ is collapsible} \implies K \text{ is contractible}.$

**Proof.** The first implication is clear: for a cone we don't have to test the apex  $e_0$  in order to see whether a set  $A$  is a face of  $K$ , since  $A \in K$  if and only if  $A \cup \{e_0\} \in K$ . The third implication is easy topology: one can write down explicit deformation retractions. The middle implication we will derive from the following lemma, which uses the notion of a *link* of a vertex  $e$  in a simplicial complex  $K$ : this is the complex  $K/e$  formed by all faces  $A \in K$  such that  $e \notin A$  but  $A \cup \{e\} \in K$ .

**Lemma 5.18.**  *$K$  is non-evasive if and only if either  $K$  is a simplex, or it is not a simplex but it has a vertex  $e$  such that both the deletion  $K \setminus e$  and the link  $K/e$  are non-evasive.*

**Proof.** If no questions need to be asked (that is, if  $c(K) = 0$ ), then  $K$  is a simplex. Otherwise we have some  $e$  that corresponds to the first question to be asked by an optimal algorithm. If one gets a YES answer, then the problem is reduced to the link  $K/e$ , since the faces  $B \in K/e$  correspond to the faces  $A = B \cup \{e\}$  of  $K$  for which  $e \in A$ . In the case of a NO-answer the problem similarly reduces to the deletion  $K \setminus e$ . 

**Proof of Theorem 5.17** ( $K$  is non-evasive  $\implies K$  is collapsible). We use induction on  $m$ , where  $m = 1$  is clear. If the vertex  $e$  corresponds to a good first vertex to ask, then we start with a sequence of collapses of the complex that correspond to a collapsing sequence for the link of  $e$  in  $K$ : this is possible by induction, since the link of  $e$  is non-evasive and has at most  $m - 1$  vertices. (A non-maximal face in the link  $K/e$  that is contained in a unique maximal face provides the same type of face in the complete complex  $K$ .) Thus we can apply collapses to  $K$  until we get that  $K/e = \{\emptyset, \{f\}\}$ . Then one further collapsing step (with  $A_0 = \{e\}$  and  $A_1 = \{e, f\}$ ) takes us to the one-point complex. 

## 5.5 A topological approach

The following trivial lemma provides the step from the topological fixed point theorems for complexes to combinatorial information.

**Lemma 5.19.** *If a (finite) group  $G$  acts vertex-transitively on a finite complex  $K$  with a fixed point, then  $K$  is a simplex.*

**Proof.** If  $V := \{v_1, \dots, v_n\}$  is the vertex set of  $K$ , then any point  $x \in K$  has a unique representation of the form

$$x = \sum_{i=1}^n \lambda_i v_i,$$


with  $\lambda_i \geq 0$  and  $\sum_{i=1}^n \lambda_i = 1$ . If the group action, with

$$gx = \sum_{i=1}^n \lambda_i gv_i,$$

is transitive, then this means that for every  $i, j$  there is some  $g \in G$  with  $gv_i = v_j$ . Furthermore, if  $x$  is a fixed point, then we have  $gx = x$  for all  $g \in G$ , and hence we get  $\lambda_i = \lambda_j$  for all  $i, j$ . From this we derive  $\lambda_i = \frac{1}{n}$  for all  $i$ . Hence we get

$$x = \frac{1}{n} \sum_{i=1}^n v_i$$

and this is a point in  $K$  only if  $K$  is the complete simplex with vertex set  $V$ .

Alternatively: the fixed point set of any group action is a subcomplex of the barycentric subdivision, by Lemma 4.2. Thus a vertex  $x$  of the fixed point complex is the barycenter of a face  $A$  of  $K$ . Since  $x$  is fixed by the whole group, so is its support, the set  $A$ . Thus vertex transitivity implies that  $A = E$ , and  $K = 2^E$ . 

**Theorem 5.20** (The Evasiveness Conjecture for prime powers: Kahn, Saks & Sturtevant [KSS84]). *All monotone non-trivial graph properties and digraph properties for graphs on a prime power number of vertices  $|V| = q = p^t$  are evasive.*

**Proof.** We identify the fixed vertex set  $V$  with  $GF(q)$ . Corresponding to a non-evasive monotone non-trivial graph property we have a non-evasive complex  $K$  on a set  $E = \binom{V}{2}$  of  $\binom{q}{2}$  vertices. By Theorem 5.17  $K$  is collapsible and hence  $\mathbb{Z}_p$ -acyclic.

The symmetry group of  $K$  includes the symmetric group  $\mathcal{S}_q$ , but we take only the subgroup of all “affine maps”

$$G := \{x \mapsto ax + b : a, b \in GF(q), a \neq 0\},$$

and its subgroup

$$P := \{x \mapsto x + b : b \in GF(q)\}$$

that permute the vertex set  $V$ , and (since we are considering graph properties) extend to an action on the vertex set  $E = \binom{V}{2}$  of  $K$ . Then we can easily verify the following facts:

- $G$  is doubly transitive on  $V$ , and hence induces a vertex transitive group of symmetries of the complex  $K$  on the vertex set  $E = \binom{V}{2}$  (interpret  $GF(q)$  as a 1-dimensional vector space, then any (ordered) pair of distinct points can be mapped to any other such pair by an affine map on the line);



- $P$  is a  $p$ -group (of order  $p^t = q$ );
- $P$  is the kernel of the homomorphism that maps  $(x \mapsto ax + b)$  to  $a \in GF(q)^*$ , the multiplicative group of  $GF(q)$ , and thus a normal subgroup of  $G$ ;
- $G/P \cong GF(q)^*$  is cyclic (this is known from your algebra class).

Taking these facts together, we have verified all the requirements of Oliver's Theorem 4.7. Hence  $G$  has a fixed point on  $K$ , and by Lemma 5.19  $K$  is a simplex, and hence the corresponding (di)graph property is trivial. 🐼

From this one can also deduce—with a lemma due to Kleitman & Kwiatkowski [KK80, Thm. 2]—that every non-trivial monotone graph property on  $n$  vertices has complexity at least  $n^2/4 + o(n^2) = m/2 + o(m)$ . (For the proof see [KSS84, Thm. 6].) This establishes the Aanderaa–Rosenberg Conjecture 5.6. On the other hand, the Evasiveness Conjecture is still an open problem for every  $n \geq 10$  that is not a prime power. Kahn, Saks & Sturtevant [KSS84, Sect. 4] report that they verified it for  $n = 6$ .

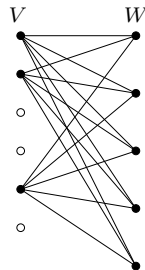
The following treats the bipartite version of the Evasiveness Conjecture. Note that in the case where  $mn$  is a prime power it follows from Proposition 5.15.

**Theorem 5.21** (The Evasiveness Conjecture for bipartite graphs, Yao [Yao88]). *All monotone non-trivial bipartite graph properties are evasive.*

**Proof.** The ground set now is  $E = V \times W$ , where any monotone bipartite graph property is represented by a simplicial complex  $K \subseteq 2^E$ .


An interesting aspect of Yao's proof is that it does not use a vertex transitive group. In fact, let the cyclic group  $G := \mathbb{Z}_n$  act by cyclically permuting the vertices in  $W$ , while leaving the vertices in  $V$  fixed. The group  $G$  satisfies the assumptions of Oliver's Theorem 4.7, with  $P = \{0\}$ . It acts on the complex  $K$  which is acyclic by Theorem 5.17. Thus we get from Oliver's Theorem that the fixed point set  $K^G$  is acyclic. This fixed point set is not a subcomplex of  $K$  (it does not contain any vertices of  $K$ ), but it is a subcomplex of the order complex  $\Delta(K)$ , which is the barycentric subdivision of  $K$  (Lemma 4.2).

The bipartite graphs that are fixed under  $G$  are those for which every vertex in  $V$  is adjacent to none, or to all, of the vertices in  $W$ ; thus they are complete bipartite graphs of the type  $K_{k,n}$  for suitable  $k$ . Our figure illustrates this for the case where  $m = 6$ ,  $n = 5$ , and  $k = 3$ .



Monotonicity now implies that the fixed graphs under  $G$  are *all* the complete bipartite graphs of type  $K_{k,n}$  with  $0 \leq k \leq r$  for some  $r$  with  $0 \leq r < m$ . (Here  $r = m$  is impossible, since then  $K$  would be a simplex, corresponding to a trivial bipartite graph property.)

Now we observe that  $K^G$  is the order complex (the barycentric subdivision) of a different complex, namely of the complex whose vertices are the complete bipartite subgraphs  $K_{1,n}$ , and whose faces are *all* sets of at most  $r$  vertices.

Thus  $K^G$  is the barycentric subdivision of the  $(r-1)$ -dimensional skeleton of an  $(m-1)$ -dimensional simplex. In particular, this space is not acyclic. Even its reduced Euler characteristic, which can be computed to be  $(-1)^{r-1} \binom{m-1}{r}$ , does not vanish. 

*Remark 5.22.* We have the following sequence of implications:

$$\text{non-evasive}^{(1)} \implies \text{collapsible}^{(2)} \implies \text{contractible}^{(3)} \implies \mathbb{Q}\text{-acyclic}^{(4)} \implies \chi = 1^{(5)},$$

which corresponds to a sequence of conjectures:

**Conjecture (k):** *Every vertex-homogeneous simplicial complex with property (k) is a simplex.*

The above implications show that

$$\text{Conj.}(5) \implies \text{Conj.}(4) \implies \text{Conj.}(3) \implies \text{Conj.}(2) \implies \text{Conj.}(1) \implies \begin{array}{l} \text{Evasiveness} \\ \text{Conjecture} \end{array}$$

Here Conjecture (5) is *true* for a prime power number of vertices, by Theorem 5.15.

However, Conjectures (5) and (4) fail for  $n = 6$ : a counterexample is provided by the six-vertex triangulation of the real projective plane (see [Mat07, Section 5.8]). Even Conjectures (3) and possibly (2) fail for  $n = 60$ : a counterexample by Oliver (unpublished), of dimension 11, is based on  $A_5$ ; see Lutz [Lut02].

So, it seems that Conjecture (1)—the monotone version of the Generalized Aanderaa–Rosenberg Conjecture 5.14—may be the right generality to prove, even though its non-monotone version fails by Proposition 5.16.

## Exercises

1. What kind of values of  $c(\mathcal{F})$  are possible for graph properties of graphs on  $n$  vertices? For monotone properties, it is assumed that one has  $c(\mathcal{F}) \in \{0, m\}$ , and this is proved if  $n$  is a prime power. In general, it is known that  $c(\mathcal{F}) \geq 2n - 4$  unless  $c(\mathcal{F}) = 0$ , by Bollobás & Eldridge [BE78], see [Bol78, Sect. VIII.5].
2. Show that the digraph property “has a sink” has complexity

$$c(\mathcal{F}_{\text{sink}}) \leq 3(n-1) - \lfloor \log_2(n) \rfloor.$$

Can you also prove that for any non-trivial digraph property one has  $c(\mathcal{F}) \geq c(\mathcal{F}_{\text{sink}})$ ?

(This is stated in Best, van Emde Boas & Lenstra [BvEBL74, p. 17]; there are analogous results by Bollobás & Eldridge [BE78] [Bol78, Sect. VIII.5] in a different model for digraphs.)

3. Show that if a complex  $K$  corresponds to a non-evasive monotone graph property, then it has a complete 1-skeleton.

4. Give examples of simplicial complexes that are contractible, but not collapsible. (The “dunce hat” is a key word for a search in the literature . . .)
5. Assume that when testing some unknown set  $A$  with respect to a set system  $\mathcal{F}$ , you always get the answer YES (unless you have already proved that the answer is NO, in which case you wouldn’t ask).
  - (i) Show that with this type of answers you *always* need  $m$  questions for *any* algorithm (and thus  $\mathcal{F}$  is evasive) if and only if  $\mathcal{F}$  satisfies the following property:
 

(\*) for any  $e \in A \in \mathcal{F}$  there is some  $f \in E \setminus A$  such that  $A \setminus \{e\} \cup \{f\} \in \mathcal{F}$ .
  - (ii) Show that for  $n \geq 5$ , the family  $\mathcal{F}$  of edge sets of planar graphs satisfies property (\*).
  - (iii) Give other examples of graph properties that satisfy (\*), and are thus evasive.

(This is the “simple strategy” of Milner & Welsh [MW76]; see Bollobás [Bol78, p. 406].)

6. Let  $\Delta$  be a vertex-homogeneous simplicial complex with  $n$  vertices and Euler characteristic  $\chi(\Delta) = -1$ . Suppose that  $n = p_1^{e_1} \cdots p_k^{e_k}$  is the prime factorization and let  $m = \max\{p_1^{e_1}, \dots, p_k^{e_k}\}$ . Prove that  $\dim \Delta \geq m - 1$ .
7. Let  $W_n^q$  be the set of all words of length  $n$  in the alphabet  $\{1, 2, \dots, q\}$ ,  $q \geq 2$ . For subsets  $\mathcal{F} \subseteq W_n^q$ , let  $c(\mathcal{F})$  be the least number of inspections of single letters (or rather, positions) that the best algorithm needs in the worst case  $s \in W_n^q$  in order to decide whether “ $s \in \mathcal{F}$ ?”

Define the polynomial

$$p_{\mathcal{F}}(x_1, \dots, x_q) = \sum_{s \in \mathcal{F}} x_1^{\mu_1} \cdots x_q^{\mu_q},$$

where  $\mu_i = \#\{j \mid s_j = i\}$  for  $s = s_1 \cdots s_q$ .

Show that

$$(x_1 + \cdots + x_q)^{n-c(\mathcal{F})} \mid p_{\mathcal{F}}(x_1, \dots, x_q)$$

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## References

- [AK92] N. Alon and D. Kleitman. Piercing convex sets and the Hadwiger Debrunner  $(p, q)$ -problem. *Adv. Math.*, 96:103–112, 1992.
- [Alo98] N. Alon. Piercing  $d$ -intervals. *Discrete Comput. Geometry*, 19:333–334, 1998.
- [Alo02] N. Alon. Covering a hypergraph of subgraphs. *Discrete Math.*, 257:249–254, 2002.
- [AZ14] M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, fifth edition, 2014.
- [BCG82] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways. Vol. 2: Games in Particular*. Academic Press, London etc., 1982.
- [BE78] B. Bollobás and S. E. Eldridge. Packings of graphs and applications to computational complexity. *J. Combinatorial Theory, Ser. B*, 25:105–124, 1978.
- [BG54] D. Blackwell and M. A. Girshick. *Theory of Games and Statistical Decisions*. Wiley, New York, 1954.
- [Bol78] B. Bollobás. *Extremal Graph Theory*. Academic Press, 1978.
- [Bre72] G. E. Bredon. *Introduction to Compact Transformation Groups*. Academic Press, New York, 1972.
- [Bre93] G. Bredon. *Topology and Geometry*. Graduate Texts in Mathematics 139. Springer-Verlag, Berlin etc., 1993.
- [Bro71] R. F. Brown. *The Lefschetz Fixed Point Theorem*. Scott, Foresman and Co., Glenview IL, 1971.
- [Bro00] C. Browne. *HEX Strategy*. A K Peters, Wellesley MA, 2000.
- [BvEBL74] M. R. Best, P. van Emde Boas, and H. W. Lenstra jr. A sharpened version of the Anderaa–Rosenberg conjecture. Technical Report ZW 30/74, Mathematisch Centrum Amsterdam, Afd. Zuivere Wisk., 1974. 20 pages.
- [dL13] M. de Longueville. *A Course on Topological Combinatorics*. Universitext. Springer, 2013.
- [Eng77] R. Engelking. *General Topology*. PWN Warszawa, 1977.
- [Fis66] G. J. Fisher. Computer recognition and extraction of planar graphs from their incidence matrix. *IEEE Transactions on Circuit Theory*, 2(CT-17):154–163, 1966.

- [Flo52] E. E. Floyd. On periodic maps and the euler characteristics of the associated spaces. *Transactions Amer. Math. Soc.*, 72:138–147, 1952.
- [Für81] Z. Füredi. Maximum degree and fractional matching in uniform hypergraphs. *Combinatorica*, 1:155–162, 1981.
- [Gal79] D. Gale. The game of Hex and the Brouwer fixed-point theorem. *Amer. Math. Monthly*, 86:818–827, 1979.
- [Gar58] M. Gardner. *The Scientific American Book of Mathematical Games and Diversions*. Simon and Schuster, New York, 1958. Reprinted in “Hexaflexagons, Probability Paradoxes, and the Tower of Hanoi”, Math. Association of America and Cambridge University Press, 2008.
- [Gar89] M. Gardner. *Mathematical Carnival*. Mathematical Assoc. Amer., Washington DC, updated and revised edition, 1989.
- [GL70] A. Gyárfás and J. Lehel. A Helly-type problem in trees. In *Combinat. Theory Appl. (P. Erdős et al. editors), Colloquia Math. Soc. Janos Bolyai 4*, pages 57–1584. North-Holland, Amsterdam, 1970.
- [GL85] A. Gyárfás and J. Lehel. Covering and coloring problems for relatives of intervals. *Discrete Math.*, 55:167–180, 1985.
- [Hel] S. Hell. On a topological fractional helly theorem. Preprint, June 2005, 11 pages, <http://arxiv.org/abs/math/0506399>.
- [HJ63] A. W. Hales and R. I. Jewett. Regularity and positional games. *Trans. Amer. Math. Soc.*, 106:222–229, 1963.
- [HT74] J. Hopcroft and R. Tarjan. Efficient planarity testing. *J. ACM*, 21:549–568, 1974.
- [Ill78] N. Illies. A counterexample to the generalized Aanderaa–Rosenberg conjecture. *Information Processing Letters*, 7:154–155, 1978.
- [JT95] T. R. Jensen and B. Toft. *Graph Coloring Problems*. Wiley-Interscience, New York, 1995.
- [Kai97] T. Kaiser. Transversals of  $d$ -intervals. *Discr. Comput. Geom.*, 18:195–203, 1997.
- [Kai98] T. Kaiser. Piercing problems and topological methods. Doctoral dissertation, Department of Applied Mathematics, Charles University, Prague, 1998.
- [KK80] D. J. Kleitman and D. J. Kwiatowski. Further results on the Aanderaa–Rosenberg conjecture. *J. Combinatorial Theory, Ser. B*, 28:85–95, 1980.

- [KR99] T. Kaiser and Yu. Rabinovich. Intersection properties of families of convex  $(n, d)$ -bodies. *Discrete Comput. Geom.*, 21:275–287, 1999.
- [KSS84] J. Kahn, M. Saks, and D. Sturtevant. A topological approach to evasiveness. *Combinatorica*, 4:297–306, 1984.
- [Lut02] F. H. Lutz. Examples of  $\mathbb{Z}$ -acyclic and contractible vertex-homogeneous simplicial complexes. *Discrete Comput. Geometry*, 27:137–154, 2002.
- [Man67] P. Mani. Zwei kombinatorische Sätze vom Typ Sperner–Tucker–Ky Fan. *Monatshefte Math. Physik*, 71:427–435, 1967.
- [Mat01] J. Matoušek. Lower bounds on the transversal numbers of  $d$ -intervals. *Discrete Comput. Geom.*, 26:283–287, 2001.
- [Mat07] J. Matoušek. *Using the Borsuk–Ulam theorem (revised 2nd printing)*. Springer, Berlin etc., 2007.
- [Meh84] Kurt Mehlhorn. *Data Structures and Efficient Algorithms. Vol. 2: Graph Algorithms and NP-Completeness*. Springer, 1984.
- [Mil95] J. Milnor. A Nobel prize for John Nash. *Math. Intelligencer*, 17(3):11–17, 1995.
- [MM96] K. Mehlhorn and P. Mutzel. On the embedding phase of the Hopcroft and Tarjan planarity testing algorithm. *Algorithmica*, 16:233–242, 1996.
- [Mun84] J. R. Munkres. *Elements of Algebraic Topology*. Addison-Wesley, Reading, MA, 1984.
- [Mun00] J. R. Munkres. *Topology. A First Course*. Prentice-Hall, Englewood Cliffs, NJ, second edition, 2000.
- [MW76] E. C. Milner and D. J. A. Welsh. On the computational complexity of graph theoretical properties. In C. S. J. A. Nash-Williams and J. Sheehan, editors, *Proc. Fifth British Combinatorial Conference, Aberdeen 1975*, Utilitas Math., pages 471–487, Winnipeg, 1976.
- [Oli75] R. Oliver. Fixed-point sets of group actions on finite acyclic complexes. *Commentarii Math. Helvetii*, 50:155–177, 1975.
- [Ram96] E. A. Ramos. Equipartition of mass distributions by hyperplanes. *Discrete Comput. Geom.*, 15:147–167, 1996.
- [Ros73] A. L. Rosenberg. On the time required to recognize properties of graphs: A problem. *SIGACT News*, 5:15–16, 1973.
- [RV76] R. Rivest and S. Vuillemin. A generalization and proof of the Aanderaa–Rosenberg conjecture. In *Proc. 7th Ann. Symp. Theory Comput., Albuquerque 1975*, pages 6–11. ACM, 1976.

- [RV78] R. Rivest and S. Vuillemin. On recognizing graph properties from adjacency matrices. *Theoretical Computer Science*, 3:371–384, 1978.
- [Sga96] J. Sgall. Solution of a covering problem related to labelled tournaments. *J. Graph Theory*, 23:111–118, 1996.
- [Smi38] P. A. Smith. Transformations of finite period. *Annals of Math.*, 39:127–164, 1938.
- [Smi41] P. A. Smith. Fixed point theorems for periodic transformations. *American J. Mathematics*, 63:1–8, 1941.
- [Spe28] E. Sperner. Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes. *Abh. Math. Sem. Hamburg*, VI:265–272, 1928.
- [Sti93] J. Stillwell. *Classical Topology and Combinatorial Group Theory*. Graduate Texts in Mathematics 72. Springer, New York, 2nd edition, 1993.
- [Tar95] G. Tardos. Transversals of 2-intervals, a topological approach. *Combinatorica*, 15:123–134, 1995.
- [Tho92] C. Thomassen. The Jordan-Schönflies theorem and the classification of surfaces. *Amer. Math. Monthly*, 99:116–130, 1992.
- [Yao88] A. C.-C. Yao. Monotone bipartite graph properties are evasive. *SIAM J. Computing*, 17:517–520, 1988.
- [Zie98] G. M. Ziegler. Shelling polyhedral 3-balls and 4-polytopes. *Discrete Comput. Geometry*, 19:159–174, 1998.